Three bodies: periodic orbits

READING LEVEL KIDS · EVERYONE · EXPERTS

Computer-assisted certificates for 29 periodic orbits of three equal masses and three at unequal masses, and a conditional KAM result for moth I; checked within this project; not yet refereed.

computer-assisted proof; the program is not published here 32 · Lean-checked 7 · certified 5 · open 5 · certified; the program is not published here 1 · conditional; the program is not published here 1

CHECK IT YOURSELF

The check guide for this book · How to check a result yourself

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What you can rerun: The exact arithmetic (stability of yin-yang II, the finite KAM conditions for moth I, distinctness of the orbits) and the Lean file are from-files. Existence, the Morse indices and the other stability verdicts are logs-only: the full files of the CAPD-based computations are too large to ship; their saved boxes and summaries are public. The Morse acceptance files are withheld for another reason, in the certificate summary's words: The acceptance files carry internal file locations, so they are not shipped; they are available on request, and their fingerprints are given here so that a copy can be matched exactly. This project's own programs are withheld for a different reason, in the check guide's words: How I direct this work, and my own tools, are not part of this book. Each result page gives its own rerun value and 5-minute check.

What a match shows: a match tells you that the published files really produce the published numbers. It doesn't tell you that the question was the right one to ask, or that a written argument around the numbers is complete. Each recipe says plainly what its steps show and what they don't. For a result whose evidence is logs (logs-only), a match checks the bookkeeping of the logs, not the search that wrote them.

Cite: Sonny (2026). Three bodies: periodic orbits (preliminary). Go check yourself. https://gocheckyourself.org/read/three-body/everyone/

1. Start here
2. What the words mean
3. The 2013 table
4. What was checked
5. Moth I
6. Sixteen more orbits
7. Statement
8. Theorems in this book
9. An ode to checking
10. Check it yourself: fifteen periodic three-body orbits, and sixteen more
11. Files and fingerprints
12. Trust and verification
13. Lean sources: how to check them
14. People and works
15. Files and rights
16. Articles

1 Start here

Sonny

What this book shows

- My result. Each of the fifteen rows of the 2013 Šuvakov–Dmitrašinović table of three equal masses is a true periodic orbit (a motion that repeats exactly), in a box narrower than 10^{-21} near the reported start, with its Morse index (what kind of path the orbit is among nearby paths: how many independent directions lead downhill) certified and its linear stability (whether small nudges grow) decided: seven rows stable, eight unstable. Certified (computer-assisted).

- Sixteen more orbits, ten from published lists and six that follow the figure-eight word (the pattern in which the path winds around the points where two stars would collide), have the same kind of existence certificate (a computer-checked proof that a true periodic orbit lies inside a stated tiny box): 29 distinct orbits in all. The orbits were found by others; the certificates are mine. Certified (computer-assisted). For moth I, the conditions of the KAM theorem (around a stable orbit that twists in the right way, there are tubes of motion arbitrarily close to it, and motion that starts on a tube stays on it for all time) hold under four stated premises. Conditional.

- Three more orbits, at unequal masses (masses 1, 1, 3/2; 1, 1, 31/16; 1, 1, 97/80), each with its own certificate: certified (computer-assisted). Three certified orbits, at masses 1, 1, 3/2; 1, 1, 31/16 and 1, 1, 97/80, are not listed at those masses in any of the 23 catalogue sets we checked. The one at 3/2 matches no row of those sets at all; the two at 31/16 and 97/80 are linearly stable and belong to families listed at other masses (mass 2, and equal masses for moth I). The article Certified orbits that the catalogues do not list gives the details.
- Why it matters. A computer search finds orbits that look periodic, but every step rounds, so a near miss can look like a return. A certificate removes that doubt for the orbit it covers; in the sources I cite, these rows were reported from numerical work.

- In Lean: one short file proves the conditional KAM implication and exact endpoint arithmetic for moth I (7 theorems, depending on no axioms). No orbit result is Lean-proved. (Lean is a proof-checking program: it checks every step of a proof against a fixed list of rules.)
- Every theorem in this book is listed in the theorem library.

The question

Three stars of equal mass pull on one another by gravity. If you know where they start and how fast they move, how do they move afterwards? Isaac Newton wrote down the law in 1687. Two bodies have an exact answer: an ellipse if they stay bound to each other, an open curve if they fly apart, a straight line if they fall head on. For three, nobody has found a general formula, and most starts give a tangle that never repeats.

The question was put in public in 1885. King Oscar II of Sweden offered a prize for it, and the announcement asked, in our paraphrase, for the coordinates of any number of mass points that attract one another by Newton's law, with no two ever colliding, written as series in known functions of time that converge uniformly. (The prize announcement of 1885 is in Acta Mathematica 7; its English wordings differ between translations, so we do not quote one. History: June Barrow-Green, Poincaré and the Three-Body

Problem, 1997.) Henri Poincaré won the prize in 1889, found an error in his own memoir, and published a corrected version in 1890. The corrected memoir showed that the motion can be chaotic, and it also showed where to look for order: in orbits that repeat exactly.

Those repeating orbits are the subject of this book. In 2013 Milovan Šuvakov and Veljko Dmitrašinović searched by computer and published a table of fifteen starting points (thirteen different orbits) for three equal masses. Their table supplies numerical starting points; interval certificates (certificates built with interval arithmetic, where every number carries a guaranteed lower and upper bound) are needed to establish a true periodic orbit for each row. The question this book answers is: is each row of that table a true periodic orbit, how does it bend (its Morse index), is it linearly stable, and for one orbit, moth I, do invariant tori (nested tubes in the space of motions; motion that starts on one stays on it) surround it? We also ask whether each is there for sixteen more orbits that other people listed from numerical searches.

The short answer is yes for existence, for all fifteen rows and all sixteen further orbits. Each lies in a stated tiny box, free of collisions. The details, with their exact labels, are below.

Why it matters

The three-body problem is the simplest case of gravity with no closed solution. Periodic orbits are the landmarks in it: they are the motions you can predict forever, and the way nearby motions behave around them (stable or unstable) shapes the whole picture. Whether an orbit is stable decides whether a configuration of three stars could last or would fly apart. A computer search finds orbits that look periodic, but every step rounds, so a near miss can look like a return. A certificate removes that doubt for the orbit it covers.

What answering it would help with

- Reference orbits with guaranteed existence give others cases to test long-time numerical integrators and validated-numerics software (software that gives guaranteed bounds on its results) against cases where the answer is known.
- The Morse index and the stability verdicts sort the many orbits found numerically into families, so that those worth deeper study can be picked out.
- The moth I result, if its four premises are examined and hold, would show invariant tori accumulating at an orbit in a system usually described as chaotic. Whether all nearby motion stays confined is open; the result gives a case for judging how often such structures occur.
- The exact boxes can serve as starting points for others who want to study these orbits with other methods or for unequal masses.

What our research shows

What this is

COMPUTER-ASSISTED CERTIFICATES, CHECKED WITH INTERVAL ARITHMETIC, THAT EACH OF THE FIFTEEN ROWS OF THE 2013 ŠUVAKOV–DMITRAŠINOVIĆ TABLE IS A TRUE PERIODIC ORBIT OF THREE EQUAL MASSES, WITH ITS MORSE INDEX AND ITS LINEAR STABILITY (7 STABLE, 8 UNSTABLE) (SONNY, 2026); THE SAME KIND OF CERTIFICATE FOR SIXTEEN MORE ORBITS THAT OTHER PEOPLE HAD LISTED FROM NUMERICAL SEARCHES (SONNY, 2026); AND, FOR MOTH I, THE CONDITIONS OF THE KAM THEOREM, UNDER FOUR STATED PREMISES (SONNY, 2026). SEPARATELY, CERTIFICATES FOR THREE ORBITS AT UNEQUAL MASSES (SONNY, 2026), EACH WITH ITS OWN ENTRY IN THE SECTION ON THEM.

Still open

KAM FOR EVERY ORBIT OTHER THAN MOTH I, AND EQUAL-MASS ORBITS OUTSIDE THE TABLES. THE 29 EQUAL-MASS ORBITS WERE FOUND BY ŠUVAKOV AND DMITRAŠINOVIĆ, BY LI AND LIAO, AND BY HRISTOV AND HRISTOVA AND THEIR CO-AUTHORS; THE CERTIFICATES ARE MINE.

Checked by

INTERVAL-ARITHMETIC CERTIFICATES; A SECOND SOLVER FOR MOTH I; A SECOND FLOW CODE (A PROGRAM THAT FOLLOWS THE MOTION STEP BY STEP), WRITTEN WITH DIFFERENT SOFTWARE, THAT CERTIFIED 9 OF THE 15 TABLE ROWS ON ITS OWN (THE OTHER 6 REST ON ONE CODE: BUTTERFLY II, BUMBLEBEE, BUTTERFLY IV, YARN, AND YIN-YANG II AT BOTH SCALES); AND A WRITTEN RECIPE FOR EACH CHECK.

Certified: certified (computer-assisted): existence for 29 distinct orbits (the 13 orbits of the 2013 table, ten orbits from published lists and six figure-eight-power orbits), Morse index for all 13 orbits of the table (all 15 rows), linear stability for 20 of the 29. Moth I KAM: accepted, conditional on four stated premises.

Three orbits at unequal masses

These three are separate from the 29 orbits above, and each is its own result. The article Certified orbits that the catalogues do not list gives the setup, the boxes and the comparison with the catalogues.

- Masses 1, 1, 3/2, zero angular momentum. Existence, absence of collisions, least period (about 9.2391238167) and uniqueness of the multiple-shooting solution (the orbit gives the only solution of the multiple-shooting equations in the full certified shooting box): certified (computer-assisted). The closest any two stars come is more than 0.00764. Linearly unstable: proved (computer-assisted); the growth factor of a small nudge over one period lies between -427.013640011164 and -427.013640011163. A second, separately written interval check of the instability agrees. An independent run of the equations by ordinary numerical integration, written separately, agrees. Comparison with 23 sets of catalogue rows: no row matches (computed). Still open: completeness of the literature.

- Masses 1, 1, 31/16. Existence, least period 12.6204700390924, closest approach more than 0.0402, and linear stability (stable): certified (computer-assisted). An independent numerical run agrees. The orbit belongs to a family that Li, Jing and Liao (2018) listed at mass 2 (their

row I.Ai.c.1 (2)); no set we checked lists mass 31/16. Still open: completeness of the literature; the stable stretch of the family, from about 1.76 to 2.10, is sampled on a grid (computed, not proved).

- Masses 1, 1, 97/80. Existence, least period 13.4210698226726, closest approach more than 0.0777, and linear stability (stable): certified (computer-assisted). The orbit belongs to the family of the 2013 row I.B.1, moth I, listed at equal masses; no set we checked lists mass 97/80. An independent run of the equations by ordinary numerical integration, written separately, agrees. Still open: completeness of the literature; the stable stretches of the family are sampled on a grid; the KAM result for moth I is at equal masses, and at mass 97/80 it is open.

The certificate files of these three orbits are in the check guide, Check 11.

What was done, in plain words

A certificate is a computer-checked proof that a true periodic orbit lies inside a stated tiny box. The plain words page, which follows on this level, explains certificates, the Morse index, linear stability and KAM, and what each does not show.

Yesterday, today, tomorrow

Yesterday. In 2013 Milovan Šuvakov and Veljko Dmitrašinović searched by computer for periodic orbits of three equal masses and published a table of fifteen starting points, which describe thirteen distinct orbits: moths, butterflies, yin-yangs and others. Later numerical studies by Li and Liao, by Dmitrašinović and colleagues, and by Hristov, Hristova and their co-authors looked at the stability of orbits of this kind and published long lists of them; those lists pointed the way to the sixteen newer orbits in chapter 4. Kapela, Zgliczyński and Simó had built interval methods that give rigorous answers about other orbits, such as the figure-eight.

Today. I gave each of the fifteen rows a certificate checked with interval arithmetic: every row's orbit exists in a box narrower than 10^{-21} near the reported start, its Morse index is certified, and its linear stability is decided (seven rows stable, eight unstable). For the Morse indices, in the certificate

summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. For moth I, the conditions of the KAM theorem hold under four stated premises. I gave sixteen more known orbits, ten from published lists and six that follow the figure-eight word, the same existence certificate (chapter 4), and seven of the ten a certified linear verdict; for the other three the test did not finish. A second flow code certified 9 of the 15 table rows on its own and agreed with all 8 linear verdicts it reached. What each recipe reruns is in the check guide.

Tomorrow. Further research is recommended on the Morse index and KAM questions for the newer orbits (the three stable Hristov-list orbits are natural candidates), on a stability verdict for the three NOT SHOWN orbits, and on the formal identity of the six figure-eight-power orbits with catalogue entries. Others may start from the exact boxes to study these orbits further, or apply the same checks to other orbits and to unequal masses. The four premises behind the moth I result are written down so that a reader can examine them, and the size of the invariant tori around moth I is left open.

Check it yourself

The certificate files, their fingerprints, the Lean file and written recipes for the main checks ship with this book; the files not shipped are available on request. Experts: the files and checks are here: check guide.

About this research: https://gocheckyourself.org/about/#how

The chapters

- The 2013 table: who found these orbits and what a certificate adds.
- What was checked: the three questions asked of each row.
- Moth I: the KAM question and its four premises.
- Sixteen more orbits: orbits from published lists, certified here.

Beside the book, the article Certified orbits that the catalogues do not list covers the three orbits at unequal masses.

The statement files are the table of results and its JSON form. The check guide has the recipes. The plain words page, the children's version and the ode to checking explain the ideas, and the credits page lists the work this stands on. The teaching clips and an interactive page are in the media folder.

We stand on the shoulders of giants: Poincaré, Šuvakov and Dmitrašinović, Li and Liao, Li, Jing and Liao, Hristov and Hristova, Kapela, Zgliczyński and Simó, and many more.

Where to go next

- Is an orbit that is "barely stable" really stable, and what would show it? See What was checked.
- If the four premises hold, what exactly do the tori around moth I tell us about the three stars? See Moth I.
- Which of the sixteen newer orbits might be next to receive a Morse index or a KAM check? See Sixteen more orbits.

Go check yourself.

Have a question about this? Ask it

2 What the words mean

What the words mean

The question: what do the four words in this book (existence, Morse index, linear stability and KAM) ask about an orbit, and what do their answers leave unsaid? The answer in brief: each word is one question about a repeating orbit of three stars; the sections below give each in plain words and in expert words.

Three stars of equal mass pull on one another by gravity. Usually the motion is a tangle. Some starting points repeat exactly: after a time T, all three stars are back where they began, moving as they began. That is a periodic orbit. In

2013 Šuvakov and Dmitrašinović found a table of them by computer. The four words below are the four questions asked of each one.

Each idea below has a short silent clip (click the picture to play it; a phone-size copy and captions are in the teaching media folder). To try the same ideas by hand, download the interactive page and open it in a browser; it works offline. The clips and the page are illustrations: the proof is the certificate.

Three stars trace moth I and come back to their dashed starting rings

What a search, a certificate and the other checks do and do not show

A numerical search follows a starting point on a computer and watches whether the stars come back to where they began. It indicates that an orbit very probably exists. Because every step rounds, it cannot rule out that a near miss looks like a return.

A certificate is a computer-checked proof that a true periodic orbit lies inside a stated tiny box. It uses interval arithmetic, in which every number carries a guaranteed lower and upper bound, so every rounding is accounted for, and a Krawczyk test, which shows that inside the larger box the computer works in, the equations for a periodic orbit have exactly one solution. The certificate establishes that periodic orbit. Stability, the Morse index and other orbits each need their own check.

The Morse index says what kind of path the orbit is among nearby paths: how many independent directions lead downhill in the action. Stability needs its own test.

Linear stability says whether small nudges grow. Its scope is the straight-line approximation of a nudge. The long-time effect of larger nudges needs a separate argument.

KAM says that, around moth I and under the four stated premises, there are invariant tubes arbitrarily close to the orbit, and motion that starts on a tube

stays on it for all time. The scope is moth I under those premises. Still open: tube sizes and the behaviour of nearby motions between the tubes.

Existence: is the orbit really there?

In plain words. A computer search gives numbers with rounding in every step, so it shows that an orbit is very likely near a starting point, not that it is there. Here the computer carries a guaranteed lower and upper bound for every number (interval arithmetic). A standard test (Krawczyk's) then shows that a true periodic orbit sits in a tiny box near the reported start, and that this orbit is the only solution of the equations the computer solved in the larger box it worked in. The same bounds show that no two stars ever collide on the way, that the period is not secretly a shorter one repeated, and that the path winds around the collision points in the pattern the 2013 table names.

Every number of moth I's start as a bracket, beside a zoom of 22 powers of ten onto the certified box

For experts. Symmetric shooting with the time-reversing symmetry R, written as multiple shooting: the zero of the map made of the node states and the return conditions x3(t), y3(t), (v1 − v2)·r1(t) is isolated in the certified shooting box by an interval Krawczyk test with C^1 enclosures from CAPD and an MPFR physical-coordinate recheck. Pair distances are bounded below on the whole enclosure, divisors of the period are excluded, and the shape-sphere crossing word is read off the enclosure rigorously.

Morse index: how does the orbit bend?

In plain words. Nature's paths make a quantity called the action stationary, like a ball resting at a point where the ground is flat. The flat point can be the bottom of a valley, the top of a hill, or a saddle. The Morse index counts how many independent directions lead downhill. It describes the shape of the action around the orbit. It is not the same as being unstable.

For experts. The index of the second variation of the action at fixed period, centre of mass fixed, counted separately in the two sectors of the reversing symmetry (plus, minus). Values depend on the function space and reduction; they are not directly comparable with indices in a choreography or symmetric loop space.

Linear stability: do small nudges grow?

In plain words. Give the stars a tiny push and watch one period. If, to a good approximation, the push only makes the motion wobble and never grow, the orbit is linearly stable. If some push grows each period, it is unstable. Of the fifteen rows, seven are stable and eight are unstable.

Yin-yang II and goggles, each beside a nudged copy, with the gap after each trip plotted below

For experts. Reduced planar linear stability: fixed energy, zero angular momentum, modulo time shift and rotation. Stable means four distinct nontrivial Floquet multipliers on the unit circle, shown with exact interval enclosures; unstable means a multiplier rigorously off the circle. Yin-yang II uses Roberts' reversible criterion on a validated 448-bit monodromy product.

Question: Give moth I a tiny push. Will a second copy, started with the push, stay close to the first, or drift away?

- It stays close.
- It drifts away.

Answer: It stays close: moth I is linearly stable, so a tiny push only makes a small wobble. An unstable orbit such as goggles does the opposite, and its copy drifts far. Try both in the figure below.

[interactive figure: see the web edition]

KAM: does it stay stable for good?

In plain words. Linear stability only looks at the straight-line approximation of a push. The full question, whether nearby motion stays close for all time, is

the question Poincaré's corrected prize memoir opened in 1890, when he found that motion near a periodic orbit can be chaotic.

Butterfly I from two starts ten decimal places apart, moving as one and then parting

The KAM theorem (Kolmogorov, Arnold, Moser) gives part of the answer. Around a stable orbit that twists in the right way, there are invisible tubes (tori) in the space of motions. Motion that starts on one of these tubes stays on it forever. There are such tubes arbitrarily close to the orbit. Motion that starts between the tubes is not covered: with three bodies, slow drift between them is not ruled out. For moth I, the conditions of the theorem have been checked, so these tubes exist around it, under four stated premises (listed in the check guide).

Dots of a Poincaré section around moth I lining up on nested closed curves, in two views

For experts. On the Poincaré return map of the reduced system (a four-dimensional real-analytic symplectic map near an elliptic fixed point), we check simple ellipticity, non-resonance through order four, a cubic Birkhoff normal form with all forbidden coefficients exactly zero, and a nondegenerate twist (det D strictly negative, about −1230.0886). Under the classical nondegenerate elliptic KAM theorem, as applied by Kapela and Simó (2017), invariant two-tori accumulate at the fixed point. The tori do not separate the four-dimensional section, so the conclusion concerns motion on the tori; Lyapunov stability needs a separate argument. The acceptance is conditional on the four premises in the check guide.

Where to go next

- Which of the four premises can a reader examine most easily? See Moth I.
- Do the sixteen newer orbits answer the same four questions? See Sixteen more orbits.

Yesterday, today, tomorrow

Yesterday: In 2013 Milovan Šuvakov and Veljko Dmitrašinović found by computer a table of repeating orbits for three equal stars, and gave them names such as moth, butterfly and yin-yang. Other people studied the stability of orbits like these with careful numerical computing. Kapela, Zgliczyński and Simó had built methods that give guaranteed answers about other orbits, such as the figure-eight.

Today: I gave all fifteen rows of the 2013 table a guaranteed answer, using interval arithmetic: each number is kept with a lower and an upper bound. All fifteen rows are really there (they describe thirteen different orbits, because some rows are the same orbit at two scales), each in a box narrower than 10^-21 near the reported start. Seven are stable and eight are unstable. For moth I, the conditions of the KAM theorem hold under four stated premises, written in the check guide. Sixteen more orbits that other people had listed from numerical searches (ten from published lists, six that follow the figure-eight word) also have certificates, and seven of the ten have a certified stability verdict; chapter 4 gives them.

Tomorrow: People may build on the exact boxes, or ask the same questions of other orbits; the Morse index and KAM questions for the newly certified orbits are open, and further research is recommended. For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. The four premises can be read and tested by anyone. How big the invariant tubes around moth I are is still an open question.

3 The 2013 table

The 2013 table

The question

Where do the known periodic orbits of three equal masses come from, and what does a certificate add to a table found by numerical search?

Why it matters

A periodic orbit found by numerical search is strong evidence that the orbit exists, but not a proof. A certificate turns that evidence into a statement anyone can check.

What answering it would help with

Certified orbits serve as fixed reference points for later work on stability and on the motion near them.

What our research shows

The answer in one line: they come from the 2013 paper of Šuvakov and Dmitrašinović, found by computer; a certificate shows by exact arithmetic that each orbit in that table is really there.

In 2013 Milovan Šuvakov and Veljko Dmitrašinović went looking for periodic orbits of three equal masses with zero angular momentum. They searched a plane of starting velocities by computer, followed each start forward, and kept the ones that came back to where they began. They then sorted what they found by the way each path winds around the collision points on the shape sphere. Their paper in Physical Review Letters gave thirteen distinct orbits as fifteen rows of initial conditions: butterflies, moths, a bumblebee, goggles, a dragonfly, yarn and yin-yangs.

That table is the foundation of chapters 1 to 3 of this family: the fifteen rows and moth I. The sixteen orbits of chapter 4 come from other published lists, credited there. There is no way to look for something if no one knows it exists. Certifying an orbit means proving that an orbit already found is really there; the finding comes before. The numerics in that search were remarkable. Finding a handful of needles in a large haystack of starting points, and naming each by its topology, is the hard, creative part.

A certificate is a different standard, not a better one. It asks the same orbit to stand up under arithmetic that brackets every rounding error, so that the conclusion holds exactly rather than to many digits. Each row of the 2013

table now has one. For every row, the published starting velocities agree with the certified box to within one unit of the last digit printed, and the certified orbit winds in the pattern the table names.

Their work gave everyone else something to build on. This family is one of those things.

Yesterday, today, tomorrow

Yesterday. In 2013 Milovan Šuvakov and Veljko Dmitrašinović searched a plane of starting velocities by computer, followed each start forward, kept the ones that came back to where they began, and sorted them by how each path winds around the collision points on the shape sphere. Their Physical Review Letters paper gave thirteen distinct orbits as fifteen rows of starting conditions.

Today. I gave every row a certificate that holds exactly, rather than to many digits. For each row the published starting velocities agree with the certified box to within one unit of the last digit printed, and the certified orbit winds in the pattern the table names. Interval arithmetic and the Krawczyk test were used, and a written recipe in the check guide lets a reader compare the certified boxes with the 2013 table.

Tomorrow. The exact boxes can serve as starting points for others who want to study these orbits with other methods. The table itself remains the reference for where the orbits are and what they are called.

Check it yourself

The comparison of the certified rows with the 2013 table is a written recipe in the check guide.

Where to go next

- Which questions were asked of each row, and which answers are firm? See What was checked.
- Why does one row, moth I, get a further question about the long-time motion around it? See Moth I.

4 What was checked

What was checked

The question

For each of the fifteen rows of the 2013 table, is the orbit really there, how does it bend, is it linearly stable, and is anything more known about the long-time motion around it?

Why it matters

Existence, bending (the Morse index) and linear stability are the basic facts about a periodic orbit. For the 2013 rows they had been reported from numerical work.

What answering it would help with

Certified values let later studies test numerical stability labels and choose orbits for further analysis, such as KAM theory.

What our research shows

The answer in one line: all fifteen rows (thirteen distinct orbits) are there; seven are linearly stable and eight unstable; moth I also has a conditional KAM result.

Each of the fifteen rows was asked three questions, and moth I a fourth. The plain words page explains each word.

1. Is the orbit there? Yes, for all fifteen rows (thirteen distinct orbits). Interval arithmetic and a Krawczyk test show that a true periodic orbit, the only solution of the shooting equations in the larger box the computer worked in, starts in a box narrower than 10^{-21} near each reported start, with collisions and shorter periods excluded and the shape word confirmed.
2. What is its Morse index? Certified for all fifteen rows, counted at fixed period in two symmetry sectors; for butterfly IV one near-singular block

needed an extra exact step, described in the check guide. For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. The values run from 8 (butterflies I and II) to 121 (butterfly IV).

3. Is it linearly stable? Decided for all fifteen: seven stable, eight unstable. Two of the stable ones are only barely so, with a cosine within 2.31×10^{-9} (yin-yang II) and 3.8×10^{-23} (butterfly IV) of 1.
4. Moth I: KAM. Accepted, conditional on four stated premises. The details are in Moth I.

The full table is in the statement of results. The figure-eight was run through the same checks as a control and gave the published answers. Its existence was proved by Chenciner and Montgomery (2000), after Moore's numerical discovery (1993); its rigorous linear stability is by Kapela and Simó (2007), and its KAM stability by Kapela and Simó (2017).

The same orbit, twice

Rows II.C.3a and II.C.3b of the 2013 table are one orbit seen at two scales: the path crosses its starting configuration twice per period with different speeds. Two separate computations at the two scales give overlapping enclosures of the scale-invariant quantity $T^2|E|^3$ (about 14017.58), and cosines that agree to about 3×10^{-45}. The 2013 paper notes the same relation.

A second code on the same rows (partial confirmation)

All fifteen rows were computed with one CAPD-based code, so a fault in that code could touch every row. As a partial check, a second flow code, based on Arb and different from CAPD, was run on the same fifteen rows. Each run started from the middle of the book's own box.

- Existence: the second code certified an orbit inside the book's box for 9 of the 15 rows, which is 8 distinct orbits (rows II.C.2a and II.C.2b are one orbit). These are butterfly I, goggles, yin-yang I (both rows), butterfly III, moth I, dragonfly, moth III and moth II.

- Linear stability: for 8 of those 9 rows the second code's verdict is the same as the book's (butterfly I, goggles, yin-yang I twice, dragonfly: unstable; butterfly III, moth I, moth II: stable). For moth III its stability test did not finish. It found no opposing verdict.
- For the other 6 rows (butterfly II, bumblebee, butterfly IV, yarn, and both scales of yin-yang II) the second code did not finish its existence test within the time given. That is marked NOT SHOWN. The existence test remains unresolved for those rows in that code.

This indicates that the nine certified rows do not depend on a fault specific to the CAPD-based code, as far as a second code can show; the rows that did not finish remain checked by one code only, and further work with the second code is recommended. The row-by-row table is in the second-computation file among the certificates. The second code's way of turning its path into a crossing word has not been reviewed, so its shape-word step is not claimed.

Sixteen more orbits

Chapter 04 applies the same questions (existence, collision exclusion, least period, and, where it finished, linear stability) to sixteen further orbits that other people had listed from numerical searches. Morse indices were not computed for them.

What you can recheck

The check guide gives a written recipe for each check. With a few lines of code in any language that has exact fractions you can recheck the table comparison, the yin-yang II stability verdict from its saved final matrices, and the finite KAM conditions for moth I.

Yesterday, today, tomorrow

Yesterday. Kapela and Zgliczyński (2003) built the interval Krawczyk method for existence of choreographies, and Kapela and Simó (2007) added rigorous linear stability and treated the figure-eight. Minton's 2013 thesis gave computer-assisted existence proofs for choreographies but does not treat these fifteen rows. Li and Liao (2014) and Dmitrašinović, Hudomal, Shibayama and Sugita (2018) examined the stability of these orbits

numerically, and their labels agree with the verdicts here. Li and Liao reran the fifteen starting points with arbitrary-precision arithmetic and report that at least seven of them depart from periodic motion after a long enough time; the certificates here are for true periodic orbits inside small boxes around the printed starting values.

Today. I asked three questions of each of the fifteen rows. Existence: all fifteen rows are there (thirteen distinct orbits), each in a box narrower than 10^-21, with collisions and shorter periods excluded and the shape word confirmed. Morse index: certified for all fifteen rows, with values from 8 to 121. For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. Linear stability: decided for all fifteen, seven stable and eight unstable, two of the stable ones only barely so. The figure-eight ran through the same checks as a control and gave the published answers, and the two scales of the same orbit (rows II.C.3a and II.C.3b) gave overlapping enclosures. A second flow code, run afterwards, certified 9 of the 15 rows, with 8 matching linear verdicts and none opposing.

Tomorrow. The second computation covers 9 of the 15 rows; running it to completion on the other 6 rows is recommended. The Morse indices are fixed-period indices in two symmetry sectors, so they may be compared with indices in other function spaces by those who work in them. The two barely stable orbits, yin-yang II and butterfly IV, may attract closer study. The partial second computation is described above.

Check it yourself

The check guide gives recipes for the table comparison and for the stability of yin-yang II. The larger computations behind the other values are stated there, not rerun.

Where to go next

- What exactly would it take to turn the moth I KAM result from conditional into unconditional? See Moth I.

- Do the checks reach beyond the 2013 table, and how far? See Sixteen more orbits.

5 Moth I

Moth I

The question

Moth I is a periodic orbit of three equal masses that is linearly stable. Does that stability survive beyond the straight-line approximation, so that motion near it stays near it?

Why it matters

Linear stability alone does not say that motion near the orbit stays near it for all time. KAM theory can, when its conditions are met.

What answering it would help with

A conditional KAM result for moth I may show which premises would have to be checked for similar results on other orbits.

What our research shows

The answer in one line: under the classical KAM theorem, invariant two-tori accumulate at the orbit, provided four stated premises hold. This is a conditional result.

Moth I (row I.B.1) has period about 14.894 and is linearly stable. The KAM question for it is whether the stability survives beyond the linear approximation.

How the question is posed

Cut the motion with a section and follow each point once around: this is the return map, here four-dimensional after the conserved quantities are removed. Moth I is a fixed point of it. The map was assembled from eight

section-to-section pieces, each computed at 384-bit precision with its derivatives to third order.

The method of Kapela and Simó (2017) then needs four facts about that map at the fixed point:

- the two rotation frequencies are simple and elliptic;
- no low-order resonance holds between them (112 small divisors, each separated from zero);
- the cubic normal form has the required shape (112 coefficients exactly zero);
- the twist is nondegenerate: a 2×2 determinant D is not zero.

The twist determinant comes out at about -1230.0886, with an interval of half-width 2.6×10^{-4}, strictly negative. Under the classical KAM theorem, invariant two-tori then accumulate at the orbit.

Status

Accepted, conditional on four premises (stated in full in the check guide):

1. The existence certificates for moth I, from this book's own existence computation of the fifteen rows, are sound.
2. The flow, derivative, interval-time and event computations of the CAPD/MPFR solver are sound.
3. The build and the solver are correct. A second implementation, using Arb with no CAPD code, puts the determinant inside the stated interval.
4. The classical KAM theorem is admitted mathematics, in the form Kapela and Simó use.

A separate native test of the solver's restart path with interval times, run against exact rational reference values, passed; it supports premise 2 and does not remove it. The certificate files for the finite conditions carry a field KAM_accepted set to false: the programs that wrote them check the finite conditions only and never apply the theorem. The acceptance above is the separate step of applying the theorem, under the four premises.

What it does and does not say

It says that there are invariant tori arbitrarily close to moth I, and that motion starting on them stays on them. It gives no size for the tori and does not rule out slow drift between them. No other row has a KAM result.

What Lean covers

A short Lean 4 file, in the lean folder of this book, states the conditional shape of the result. It proves that if seven stated hypotheses hold (a fixed point, analyticity, a symplectic map, elliptic spectrum, no low-order resonance, the required normal form, a nondegenerate twist) and a KAM criterion is supplied as an argument, then invariant tori follow. It also proves that the twist determinant alone does not supply the other six hypotheses, and it checks exact endpoint arithmetic on the twist determinant (positive denominators, endpoint order, a negative upper endpoint and two outward bounds). The file has 12 declarations (a structure, four definitions and seven theorems); the seven theorems use no axioms, and the file builds standalone with the Lean version pinned beside it. Lean does not check the KAM theorem (it is an argument, not an axiom), the numerical values (they are copied in as integers), the link between the saved files and those integers, or any ODE enclosure. Those remain the four premises.

Yesterday, today, tomorrow

Yesterday. The KAM theorem of Kolmogorov, Arnold and Moser says that around a stable orbit that twists in the right way there are invariant tubes. Kapela and Simó (2017) showed how to check its conditions around an arbitrary periodic orbit, and Wilczak and Barrio (2017) gave a published precedent for combining rigorous derivatives with a KAM-type theorem. Dmitrašinović and colleagues had found moth I linearly stable by numerical computation.

Today. I applied the method of Kapela and Simó to moth I and checked the four facts it needs: simple elliptic rotation frequencies, no low-order resonance (112 small divisors, each separated from zero), the required cubic normal form (112 coefficients exactly zero), and a nondegenerate twist (determinant

about -1230.0886, strictly negative). The result is that invariant two-tori accumulate at the orbit, conditional on four premises: the existence certificates, the solver contracts, the build and solver, and the classical KAM theorem. A second implementation with no CAPD code puts the twist determinant inside the stated interval. A short Lean file checks the conditional implication and the endpoint arithmetic. No other row has a KAM result.

Tomorrow. The four premises are stated so that readers can examine each one, and the same conditions could be checked for other stable rows; further research is recommended. Sizes of the tori, and whether slow drift between them can occur, are not covered and stay open.

Check it yourself

The finite KAM conditions for moth I are a written recipe in the check guide, and the short Lean file is in the lean folder of this book.

Where to go next

- Do the other stable rows have the same twist and normal form, so that a KAM check could be run for them? See What was checked.
- Do the three stable orbits from the Hristov list, among the newer orbits, look like candidates for the same check? See Sixteen more orbits.

6 Sixteen more orbits

Sixteen more orbits

The question

Beyond the 2013 table, other people have listed many periodic orbits of three equal masses from numerical searches. Are sixteen of them really there, and which are linearly stable?

Why it matters

Published lists of orbits from numerical searches are much longer than the list of orbits with certificates.

What answering it would help with

Certificates for listed orbits may let others build on them with the same confidence as on the 2013 rows.

What our research shows

The answer in one line: all sixteen exist, each in a stated tiny box and free of collisions; seven of the ten orbits from published lists have a certified stability verdict (three stable, four unstable), and three are NOT SHOWN.

Chapters 1 to 3 cover the fifteen rows of the 2013 table. This chapter adds sixteen more periodic orbits of three equal masses. All sixteen are orbits that other people had already found by numerical search and listed. What this book adds is a certificate for each one: a computer-checked statement that a true periodic orbit lies inside a stated tiny box.

Three further orbits, at unequal masses, are not among these sixteen. They are in the article Certified orbits that the catalogues do not list: one at masses 1, 1, 3/2 and two linearly stable ones at masses 1, 1, 31/16 and 1, 1, 97/80. None of the three is listed at its masses in any of the 23 catalogue sets we checked; the two stable ones belong to families listed at other masses.

What the words mean, and what they do not show

Here is, in plain words, what has and has not been done. The plain words page says the same at more length.

- A numerical search follows a starting point on a computer and sees whether the stars come back near where they began. It indicates that an orbit very probably exists. It cannot exclude that rounding errors made a near miss look like a return.

- A certificate is a computer-checked proof that a true periodic orbit lies inside a stated tiny box. Two tools do the work. Interval arithmetic keeps every number as a guaranteed lower and upper bound, so every rounding is accounted for. A Krawczyk test then shows that, inside the larger box the computer works in, the equations for a periodic orbit have exactly one solution. The same bounds show that no two stars collide along the path and that the period is not a shorter period repeated. A certificate says the orbit is there. It says nothing by itself about stability, about the Morse index, or about other orbits.

- The Morse index says what kind of path the orbit is among nearby paths: it counts the independent directions in which the action goes downhill. Stability needs its own test.
- Linear stability says whether small nudges grow. If, in the straight-line approximation, a nudge only wobbles, the orbit is linearly stable; if some nudge grows each period, it is unstable. The long-time effect of larger nudges needs a separate argument.
- KAM (Kolmogorov, Arnold, Moser) is a theorem about the long-time motion near a stable orbit. For moth I, under the four premises listed in the check guide, it gives invariant tubes that come arbitrarily close to the orbit; motion that starts on a tube stays on it for all time. Still open: tube sizes and the behaviour of nearby motions between the tubes.

This book says "certified (computer-assisted)" for what it has shown. What this chapter certifies for the sixteen orbits is existence, absence of collisions, and least period for all sixteen, and linear stability for seven of the ten orbits from published lists. It does not give a Morse index or a KAM statement for any of the sixteen, and it does not replace checking by others, which we invite (see the ode to checking).

Credit

We stand on the shoulders of giants. The directions came from other people's published lists. Xiaoming Li and Shijun Liao published numerical orbits in 2017. Ivan Hristov, Radoslava Hristova and their co-authors published long lists of numerical initial conditions, including a list of 971 numerically stable

orbits (with Kiyotaka Tanikawa) and an Euler-type list with 421,562 entries. Their published lists pointed the way to these orbits. The certificates here are our own computation. Their data files are not redistributed in this book, and rights in them remain with the authors; where a row below names a list entry, it is a reference, and the numbers in the certificates are our own certified boxes. Full references are in the credits page.

Ten orbits from published lists

Each box below is an exact rational box in the 2013 coordinates (equal masses, positions (-1, 0), (1, 0), (0, 0), velocities (vx, vy), (vx, vy), -2(vx, vy)): about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period. The published period is a rounded locator for the list entry; it is not a certified number. The exact boxes are in the ten-orbit file among the certificates; the saved matrices of the existence test for Li-Liao I.A_3 are shipped, and the check guide gives the recipe for rechecking them.

Label as the source gives it	**Published period (rounded)**	**Pair distance, lower bound**	**Linear stability**
Hristov list #5	20.749065	0.00539	stable (barely: a cosine root within 4 $\times 10^{-5}$ of 1)
Li-Liao I.B_7 (Euler #2023)	33.641419	0.0241	unstable
Li-Liao I.A_3 (Euler #4)	37.320524	0.493	unstable
Hristov list #9	46.066900	0.0289	stable
Li-Liao II.C_2 (Euler #64)	54.750191	0.0274	unstable
Hristov list #10	73.740125	0.403	stable
Hristov list #29	98.649460	0.00269	NOT SHOWN
Li-Liao II.C_61 (Euler #96)	111.673551	0.345	unstable
Hristov list #67	124.536537	0.370	NOT SHOWN
Hristov list #87	251.991317	0.436	NOT SHOWN

For all ten, the certificate gives existence of a periodic orbit in the box, and that it is the only solution of the shooting equations in the larger certified box, no collisions, a least period equal to the box period, and a crossing word that matches the published word. Linear stability is certified for seven: three stable and four unstable. For Hristov lists #29, #67 and #87 the stability test did not finish, and NOT SHOWN means exactly that: it is neither a stability nor an instability finding.

The numbers behind the stability verdicts, the derivative factors of the flow, are among the certificates, and the check guide gives the recipe for recomputing each verdict from them by exact rational interval arithmetic.

Six known orbits, certified here

These six are orbits whose path follows the figure-eight word repeated many times. They are known orbits, certified here: each of the six was compared with the published catalogues, and an apparent match was found for all six. We name each by the number of repeats and its period.

Orbit	Figure-eight word repeated k times	Period (rounded)	Pair distance, lower bound	Apparent match in Hristov and Hristova's Euler list (row)	Apparent match in the 821-satellite list (row)
17 repeats, period 116.22	17	116.220683	0.553779	260854	190
18 repeats, period 85.49	18	85.493542	0.290206	295352	205
11 repeats, period 55.91	11	55.913729	0.285147	72771	85
10 repeats, period 77.42	10	77.424620	0.234238	49771	59
9 repeats, period 74.73	9	74.734329	0.179527	30084	none in that file
10 repeats, period 72.92	10	72.916266	0.147755	49787	none in that file

For each, the certificate gives existence, no collisions and least period, in a box about 2×10^{-23} wide in each velocity and 4×10^{-23} in the period. Each is a different orbit from the figure-eight run k times, from the other five, and from the other orbits in this book (the check guide gives the test). A separate exact test at one third of the period rules out an equal-phase choreography for five of the six; for the orbit with 10 repeats and period 77.42 that test does not decide, which is marked NOT SHOWN. The numerical multipliers of all six lie far from the unit circle (largest moduli between about 4 and 87,000), which indicates that they are linearly unstable; these are computed numbers, so their linear stability is NOT SHOWN (multipliers COMPUTED only) and no stability result is claimed. The boxes and the third-of-a-period tests are among the certificates.

The catalogue matches are numerical: the start velocities, period and scale-free period agree with the published row to the printed digits, and four of the six also agree with a row of the 821-satellite list. These matches indicate that the six are the orbits listed there. A formal statement that a certified orbit is the catalogue entry is not available, and further research is recommended. We did not obtain the earliest publication history of each entry.

What we did not do

- No Morse index and no KAM statement for any of the sixteen. The three stable orbits from the Hristov list (#5, #9, #10) are natural candidates for a KAM check; whether the conditions hold has not been tested, and we recommend further study.
- The stability test for Hristov lists #29, #67 and #87 did not finish.
- The six figure-eight-power orbits have no certified linear stability: NOT SHOWN, multipliers COMPUTED only.
- The recipes in the check guide take the flow enclosures as given; they do not integrate the equations again.
- The large matrices behind the existence test (12 to 73 megabytes per orbit) are shipped for two orbits only, as worked examples that the check guide shows how to recheck; for the other fourteen they are available on request.

- We did not search the interval-arithmetic literature for earlier certificates of these particular sixteen orbits. In the sources we checked, we found no earlier certificate of them: Li and Liao (2017), the Hristov and Hristova Euler list, the 821-satellite database, the Hristov, Hristova and Tanikawa list of 971 stable orbits, Šuvakov (2014), Dmitrašinović and colleagues (2018), and Simó's list of 345 choreographies. These are numerical catalogues and papers, which would not be expected to contain certificates, so this is weak evidence. A systematic search of the rigorous-numerics literature is recommended before anyone concludes that no earlier certificate exists.

Yesterday, today, tomorrow

Yesterday. Li and Liao (2017) searched numerically for more families of equal-mass orbits. Hristov and Hristova (2024) and Hristov, Hristova and Tanikawa (2026) published long numerical lists, including lists of stable orbits. Šuvakov (2014) listed an approximate orbit that looks like the orbit with 11 repeats and period 55.91; we regard that identification as conjectured. Kapela, Zgliczyński and Simó had shown how to certify orbits of this kind with interval arithmetic.

Today. I certified ten orbits taken from the published lists and six more known orbits that follow the figure-eight word, sixteen in all: each lies in a stated box, is free of collisions, and has the stated least period. Seven of the ten have a certified linear verdict (three stable, four unstable). With the fifteen rows of the 2013 table, 29 distinct orbits now have certificates here. The end-to-end checker certifies all sixteen inside the book's boxes.

Tomorrow. The boxes can be used by others who want to study these orbits further. The Morse index and KAM questions for the three stable orbits from the Hristov list, a stability verdict for the three NOT SHOWN rows, and a search of the rigorous-numerics literature are all recommended further work.

Check it yourself

The check guide has recipes for the sixteen boxes, for seven of the ten linear verdicts and for the distinctness of all 30 orbits (the 29 certified here and the

figure-eight).

Where to go next

- Could the three stable orbits from the Hristov list carry a KAM result like moth I? See Moth I.
- How were the fifteen table rows checked, and what do the sixteen lack by comparison? See What was checked.

7 Statement

Statement

The question: for each row of the 2013 table of periodic three-body orbits, and for sixteen more orbits, is the orbit really there, how does it bend, and is it linearly stable? The answer: yes for existence in every case; the tables below give each Morse index, stability verdict and KAM status with its exact label.

Setup: Equal masses 1, G = 1. Positions x1 = (-1, 0), x2 = (1, 0), x3 = (0, 0). Velocities v1 = v2 = (vx, vy), v3 = -2 (vx, vy). Planar, zero angular momentum. Rationals are exact endpoints.

Source of the rows: M. Šuvakov and V. Dmitrašinović, Phys. Rev. Lett. 110, 114301 (2013), Table I. The exact certified boxes are in the existence boxes file among the certificates.

#	Orbit	2013 label	2013 start (vx, vy)	Existence	Morse index (plus + minus)	Linear stability	KAM
known case	figure-eight	known case	-	certified	$2 = 1 + 1$; in the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks	stable	earlier published work (Kapela and Simó), not claimed here
1	butterfly I	I.A.1	0.30689, 0.12551	certified	$8 = 4 + 4$	unstable	not claimed
2	butterfly II	I.A.2	0.39295, 0.09758	certified	$8 = 3 + 5$	unstable	not claimed
3	bumblebee	I.A.3	0.18428, 0.58719	certified	$43 = 21 + 22$	stable	not claimed
4	moth I	I.B.1	0.46444, 0.39606	certified	$9 = 4 + 5$	stable	accepted, conditional on four stated premises (listed in the check guide)
5	moth II	I.B.2	0.43917, 0.45297	certified	$17 = 8 + 9$	stable	not claimed
6	butterfly III	I.B.3	0.40592, 0.23016	certified	$15 = 7 + 8$	stable	not claimed
7	moth III	I.B.4	0.38344, 0.37736	certified	$23 = 11 + 12$	unstable	not claimed
8	goggles	I.B.5	0.08330, 0.12789	certified	$17 = 9 + 8$	unstable	not claimed
9	butterfly IV	I.B.6	0.350112, 0.07934	certified	$121 = 60 + 61$	stable	not claimed
10	dragonfly	I.B.7	0.08058, 0.58884	certified	$13 = 7 + 6$	unstable	not claimed
11	yarn	II.B.1	0.55906, 0.34919	certified	$32 = 16 + 16$	unstable	not claimed

#	Orbit	2013 label	2013 start (vx, vy)	Existence	Morse index (plus + minus)	Linear stability	KAM
12	yin-yang I	II.C.2a	0.51394, 0.30474	certified	$12 = 6 + 6$	unstable	not claimed
13	yin-yang I	II.C.2b	0.28270, 0.32721	certified	$12 = 6 + 6$	unstable	not claimed
14	yin-yang II	II.C.3a	0.41682, 0.33033	certified	$53 = 26 + 27$	stable	not claimed
15	yin-yang II	II.C.3b	0.41734, 0.31310	certified	$53 = 26 + 27$	stable	not claimed

Counts: existence 15 of 15, Morse index certified for 15 of 15, linear stability 15 of 15 (7 stable, 8 unstable). In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks. For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks. For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. Rows II.C.2a/2b and II.C.3a/3b are one orbit each at two scales, so the fifteen rows are thirteen distinct orbits. The figure-eight is a known case; its existence, stability and KAM result are earlier published work, not claims here.

Each statement says what its certificate shows: the stated box contains the starting velocities and period of a periodic orbit, and that orbit gives the only solution of the multiple-shooting equations in the full certified shooting box. Scope and premises are in the check guide.

Sixteen more orbits

Same setup. These are orbits that other people had listed from numerical searches; the certificates are Sonny's (2026). Boxes are in the ten-orbit file and in the figure-eight-power folders among the certificates. "Certified"

means certified by computer assistance (interval arithmetic and a Krawczyk test) for existence in the box, absence of collisions and least period. Morse index and KAM were not computed for any of them.

Ten orbits from published lists

Label as the source gives it	Published period (rounded)	Existence	Linear stability
Hristov list #5	20.749065	certified	stable
Li-Liao I.B_7 (Euler #2023)	33.641419	certified	unstable
Li-Liao I.A_3 (Euler #4)	37.320524	certified	unstable
Hristov list #9	46.066900	certified	stable
Li-Liao II.C_2 (Euler #64)	54.750191	certified	unstable
Hristov list #10	73.740125	certified	stable
Hristov list #29	98.649460	certified	NOT SHOWN
Li-Liao II.C_61 (Euler #96)	111.673551	certified	unstable
Hristov list #67	124.536537	certified	NOT SHOWN
Hristov list #87	251.991317	certified	NOT SHOWN

Six known orbits, certified here

Orbit	Figure-eight word repeated k times	Existence	Linear stability	Equal-phase choreography excluded at T/3	Apparent catalogue match (Euler row; 821-satellite row)
17 repeats, period 116.22	17	certified	NOT SHOWN (multipliers COMPUTED only)	yes	260854; 190
18 repeats, period 85.49	18	certified	NOT SHOWN (multipliers COMPUTED only)	yes	295352; 205
11 repeats, period 55.91	11	certified	NOT SHOWN (multipliers COMPUTED only)	yes	72771; 85
10 repeats, period 77.42	10	certified	NOT SHOWN (multipliers COMPUTED only)	NOT SHOWN	49771; 59
9 repeats, period 74.73	9	certified	NOT SHOWN (multipliers COMPUTED only)	yes	30084; none
10 repeats, period 72.92	10	certified	NOT SHOWN (multipliers COMPUTED only)	yes	49787; none

Counts for the equal-mass orbits: 29 distinct orbits with certified existence (13 in the 2013 table, 10 and 6 here), plus the figure-eight known case; 32 certified boxes in all, 30 pairwise distinct orbits including the figure-eight. Morse index certified for 13 of 29 (all 15 table rows). In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks. For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks. For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. Linear stability certified for 20 of 29 (9 stable, 11 unstable); 9 NOT SHOWN (3 orbits from published lists whose test did not

finish, and the 6 figure-eight-power orbits, whose multipliers are COMPUTED only). KAM: moth I only, conditional on four stated premises.

Apparent catalogue matches are numerical agreement with a published row; a formal identification of a certified orbit with a catalogue entry is OPEN, and further research is recommended.

Second computation of the 2013 rows: 9 of 15 rows (8 distinct orbits) certified inside the book's boxes by a second flow code; 8 linear verdicts agree and none opposes; 6 rows NOT SHOWN, which is not a disproof (row table: the second-computation file among the certificates).

Three orbits at unequal masses

A different setup: masses 1, 1 and the third mass given in the Masses column, $G = 1$. These are Sonny's (2026), each its own result, and they are not counted in the 29 above. Their setup, boxes and catalogue comparison are in the article Certified orbits that the catalogues do not list. The certificate files of these three orbits are in the check guide, Check 11.

Masses	Existence, no collisions, least period	Least period	Closest approach, lower bound	Linear stability	Comparison with the catalogues	Still open
1, 1, 3/2 (zero angular momentum)	certified, with uniqueness of the multiple-shooting solution in the full certified shooting box	about 9.2391238167	0.00764	unstable: proved (computer-assisted); growth factor between -427.013640011164 and -427.013640011163	not in any of the 23 sets of rows we checked (computed)	completeness of the literature
1, 1, 31/16	certified	12.6204700390924	0.0402	stable: certified (computer-assisted)	in a family that Li, Jing and Liao (2018) listed at mass 2 (row I.Ai.c.1 (2)); no set we checked lists mass 31/16	completeness of the literature; the stable stretch of the family (about 1.76 to 2.10) is sampled on a grid
1, 1, 97/80	certified	13.4210698226726	0.0777	stable: certified (computer-assisted)	in the family of the 2013 row I.B.1, moth I	completeness of the literature; the stable stretches of the family are sampled on a grid; KAM at this mass

Where to go next

- How do the four premises behind the moth I entry come about? See Moth I.
- Where do the sixteen further orbits come from? See Sixteen more orbits.

8 Theorems in this book

Theorems in this book

Every theorem and lemma in this book, with its exact statement, status and credit. Maths is written in LaTeX notation, between dollar signs.

The entries follow the book's own order: the figure-eight control and the fifteen rows of the 2013 table, as in the statement page; then the ten orbits from published lists and the six figure-eight-power orbits of chapter 4; then the distinctness of the orbits, the second computation, the moth I KAM result and the seven Lean theorems; then three orbits at unequal masses (Theorems 44 to 46), from the article Certified orbits that the catalogues do not list. The Lean names are internal labels, not terms of the problem.

Setup used by every orbit entry except Theorems 44 to 46, which state their own masses: equal masses 1, $G = 1$; positions $x_1 = (-1, 0)$, $x_2 = (1, 0)$, $x_3 = (0, 0)$; velocities $v_1 = v_2 = (v_x, v_y)$ and $v_3 = -2(v_x, v_y)$; planar, zero angular momentum. Rationals are exact endpoints. "Certified" means certified by computer assistance (interval arithmetic and a Krawczyk test). Each statement says what its certificate shows: the stated box contains the starting velocities and period of a periodic orbit, and that orbit gives the only solution of the multiple-shooting equations in the full certified shooting box.

Proposition 1: the figure-eight, run as a control (known case; earlier published work)

Credit: Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578. It is not a claim of this book. This result (Sonny): a check of another's result (verdict: Morse index certified) (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

The figure-eight orbit of three equal masses (the setup above) has Morse index $2 = 1 + 1$ (plus + minus) and is linearly stable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks. It was run through the same

checks as a control and gave the published answers. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: Morse index: certified (computer-assisted); existence, stability and KAM result: earlier published work, not claimed here. For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement), line "known case"; chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); boxes in certificates/existence/boxes.json.

Theorem 2: butterfly I (I.A.1), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.30689, 0.12551)$ (2013 label I.A.1, 2013 period 6.2356) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $8 = 4 + 4$ (plus + minus), and it is linearly unstable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 3: butterfly II (I.A.2), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.39295, 0.09758)$ (2013 label I.A.2, 2013 period 7.0039) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $8 = 3 + 5$ (plus + minus), and it is linearly unstable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 4: bumblebee (I.A.3), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.18428, 0.58719)$ (2013 label I.A.3, 2013 period 63.5345) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $43 = 21 + 22$ (plus + minus), and it is linearly stable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 5: moth I (I.B.1), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability; the KAM method of Kapela and Simó (2017), for the KAM conditions. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint; GNU MPFR and GNU GMP. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.46444, 0.39606)$ (2013 label I.B.1, 2013 period 14.8939) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $9 = 4 + 5$ (plus + minus), and it is linearly stable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: accepted, conditional on four stated premises (listed in the check guide).

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 6: moth II (I.B.2), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.43917, 0.45297)$ (2013 label I.B.2, 2013 period 28.6703) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $17 = 8 + 9$ (plus + minus), and it is linearly stable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 7: butterfly III (I.B.3), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.40592, 0.23016)$ (2013 label I.B.3, 2013 period 13.8658) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $15 = 7 + 8$ (plus + minus), and it is linearly stable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 8: moth III (I.B.4), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.38344, 0.37736)$ (2013 label I.B.4, 2013 period 25.8406) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $23 = 11 + 12$ (plus + minus), and it is linearly unstable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 9: goggles (I.B.5), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.08330, 0.12789)$ (2013 label I.B.5, 2013 period 10.4668) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $17 = 9 + 8$ (plus + minus), and it is linearly unstable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 10: butterfly IV (I.B.6), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-27} near the 2013 start $(v_x, v_y) = (0.350112, 0.07934)$ (2013 label I.B.6, 2013 period 79.4759) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within about 10^{-28}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $121 = 60 + 61$ (plus + minus), and it is linearly stable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 11: dragonfly (I.B.7), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.08058, 0.58884)$ (2013 label I.B.7, 2013 period 21.2710) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $13 = 7 + 6$ (plus + minus), and it is linearly unstable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 12: yarn (II.B.1), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.55906, 0.34919)$ (2013 label II.B.1, 2013 period 55.5018) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $32 = 16 + 16$ (plus + minus), and it is linearly unstable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 13: yin-yang I (II.C.2a), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.51394, 0.30474)$ (2013 label II.C.2a, 2013 period 17.3284) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $12 = 6 + 6$ (plus + minus), and it is linearly unstable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks. Rows II.C.2a and II.C.2b are one orbit at two scales, and rows II.C.3a and II.C.3b are one orbit at two scales. In the check guide's words, existence, the Morse

indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 14: yin-yang I (II.C.2b), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.28270, 0.32721)$ (2013 label II.C.2b, 2013 period 10.9626) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $12 = 6 + 6$ (plus + minus), and it is linearly unstable. In the check guide's words, existence, the Morse indices and the other stability

verdicts are stated here and are not recomputed by these checks. Rows II.C.2a and II.C.2b are one orbit at two scales, and rows II.C.3a and II.C.3b are one orbit at two scales. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 15: yin-yang II (II.C.3a), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.41682, 0.33033)$ (2013 label II.C.3a, 2013 period 55.7898) contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations;

collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $53 = 26 + 27$ (plus + minus), and it is linearly stable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks. Rows II.C.2a and II.C.2b are one orbit at two scales, and rows II.C.3a and II.C.3b are one orbit at two scales. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 16: yin-yang II (II.C.3b), a periodic orbit with its Morse index and linear stability

Credit: Builds on: the orbit, found numerically by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: CAPD::DynSys, by Tomasz Kapela, Marian Mrozek, Daniel Wilczak and Piotr Zgliczyński (2021), Communications in Nonlinear Science and Numerical Simulation 101, 105578; Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, Morse index and linear stability certified (for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists).

For the setup above, a certified box narrower than 10^{-21} near the 2013 start $(v_x, v_y) = (0.41734, 0.31310)$ (2013 label II.C.3b, 2013 period 54.2076) contains the starting velocities and period of a periodic orbit. The full

certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-22}, and in that full box this orbit gives the only solution of the multiple-shooting equations; collisions and shorter periods are excluded and the shape word is confirmed. Its Morse index is $53 = 26 + 27$ (plus + minus), and it is linearly stable. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks. Rows II.C.2a and II.C.2b are one orbit at two scales, and rows II.C.3a and II.C.3b are one orbit at two scales. In the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks.

Status: existence, Morse index and linear stability: certified (computer-assisted). For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. KAM: not claimed.

Proof: statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement); chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); exact box in certificates/existence/boxes.json; Morse evidence in certificates/morse/; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 17: the orbit Hristov list #5, existence and linear stability

Credit: Builds on: the orbit, listed from numerical searches by others (Hristov, Hristova and Tanikawa stable list); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence. Tools: Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, absence of collisions and least period certified; linear stability certified.

For the setup above, an exact rational box about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period, around the orbit with published period (rounded locator, not a certified number) 20.749065, contains the starting

velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.00539), the least period equals the box period, and the crossing word matches the published word. It is linearly stable (barely: a cosine root within 4×10^{-5} of 1). No Morse index and no KAM statement is made for this orbit.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: certified (computer-assisted) (stable). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/published-lists/published-list-orbits.json; Floquet data in certificates/published-lists/floquet/hristov-5.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 18: the orbit Li-Liao I.B_7 (Euler #2023), existence and linear stability

Credit: Builds on: the orbit, listed from numerical searches by others (Li and Liao 2017; Hristov-Hristova Euler list); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence. Tools: Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, absence of collisions and least period certified; linear stability certified.

For the setup above, an exact rational box about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period, around the orbit with published period (rounded locator, not a certified number) 33.641419, contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.0241), the least period equals the box period, and the

crossing word matches the published word. It is linearly unstable. No Morse index and no KAM statement is made for this orbit.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: certified (computer-assisted) (unstable). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/published-lists/published-list-orbits.json; Floquet data in certificates/published-lists/floquet/li-liao-i-b7.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 19: the orbit Li-Liao I.A_3 (Euler #4), existence and linear stability

Credit: Builds on: the orbit, listed from numerical searches by others (Li and Liao 2017; Hristov-Hristova Euler list); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence. Tools: Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, absence of collisions and least period certified; linear stability certified.

For the setup above, an exact rational box about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period, around the orbit with published period (rounded locator, not a certified number) 37.320524, contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.493), the least period equals the box period, and the crossing word matches the published word. It is linearly unstable. No Morse index and no KAM statement is made for this orbit.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: certified (computer-assisted) (unstable). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/published-lists/published-list-orbits.json; Floquet data in certificates/published-lists/floquet/li-liao-i-a3.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 20: the orbit Hristov list #9, existence and linear stability

Credit: Builds on: the orbit, listed from numerical searches by others (Hristov, Hristova and Tanikawa stable list); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence. Tools: Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, absence of collisions and least period certified; linear stability certified.

For the setup above, an exact rational box about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period, around the orbit with published period (rounded locator, not a certified number) 46.066900, contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.0289), the least period equals the box period, and the crossing word matches the published word. It is linearly stable. No Morse index and no KAM statement is made for this orbit.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: certified (computer-assisted) (stable). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/published-lists/published-list-orbits.json; Floquet data in certificates/published-lists/floquet/hristov-9.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 21: the orbit Li-Liao II.C_2 (Euler #64), existence and linear stability

Credit: Builds on: the orbit, listed from numerical searches by others (Li and Liao 2017; Hristov-Hristova Euler list); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence. Tools: Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, absence of collisions and least period certified; linear stability certified.

For the setup above, an exact rational box about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period, around the orbit with published period (rounded locator, not a certified number) 54.750191, contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.0274), the least period equals the box period, and the crossing word matches the published word. It is linearly unstable. No Morse index and no KAM statement is made for this orbit.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: certified (computer-assisted) (unstable). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/published-lists/published-list-orbits.json; Floquet data in certificates/published-lists/floquet/li-liao-ii-c2.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 22: the orbit Hristov list #10, existence and linear stability

Credit: Builds on: the orbit, listed from numerical searches by others (Hristov, Hristova and Tanikawa stable list); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence. Tools: Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, absence of collisions and least period certified; linear stability certified.

For the setup above, an exact rational box about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period, around the orbit with published period (rounded locator, not a certified number) 73.740125, contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.403), the least period equals the box period, and the crossing word matches the published word. It is linearly stable. No Morse index and no KAM statement is made for this orbit.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: certified (computer-assisted) (stable). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/published-lists/published-list-orbits.json; Floquet data in certificates/published-lists/floquet/hristov-10.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 23: the orbit Hristov list #29, existence

Credit: Builds on: the orbit, listed from numerical searches by others (Hristov, Hristova and Tanikawa stable list); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence. Tools: Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, absence of collisions and least period certified.

For the setup above, an exact rational box about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period, around the orbit with published period (rounded locator, not a certified number) 98.649460, contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair

distance lower bound 0.00269), the least period equals the box period, and the crossing word matches the published word. Linear stability is NOT SHOWN: the stability test did not finish, and NOT SHOWN is neither a stability nor an instability finding. No Morse index and no KAM statement is made for this orbit.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: NOT SHOWN. Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/published-lists/published-list-orbits.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 24: the orbit Li-Liao II.C_61 (Euler #96), existence and linear stability

Credit: Builds on: the orbit, listed from numerical searches by others (Li and Liao 2017; Hristov-Hristova Euler list); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence. Tools: Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, absence of collisions and least period certified; linear stability certified.

For the setup above, an exact rational box about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period, around the orbit with published period (rounded locator, not a certified number) 111.673551, contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.345), the least period equals the box period, and the crossing word matches the published word. It is linearly unstable. No Morse index and no KAM statement is made for this orbit.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: certified (computer-assisted) (unstable). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/published-lists/published-list-orbits.json; Floquet data in certificates/published-lists/floquet/li-liao-ii-c61.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 25: the orbit Hristov list #67, existence

Credit: Builds on: the orbit, listed from numerical searches by others (Hristov, Hristova and Tanikawa stable list); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence. Tools: Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, absence of collisions and least period certified.

For the setup above, an exact rational box about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period, around the orbit with published period (rounded locator, not a certified number) 124.536537, contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.370), the least period equals the box period, and the crossing word matches the published word. Linear stability is NOT SHOWN: the stability test did not finish, and NOT SHOWN is neither a stability nor an instability finding. No Morse index and no KAM statement is made for this orbit.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: NOT SHOWN. Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-

sixteen-more-orbits); certificates/published-lists/published-list-orbits.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 26: the orbit Hristov list #87, existence

Credit: Builds on: the orbit, listed from numerical searches by others (Hristov, Hristova and Tanikawa stable list); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence. Tools: Arb (Fredrik Johansson) through python-flint. This result (Sonny): existence, absence of collisions and least period certified.

For the setup above, an exact rational box about 2×10^{-25} wide in each velocity and 4×10^{-25} in the period, around the orbit with published period (rounded locator, not a certified number) 251.991317, contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.436), the least period equals the box period, and the crossing word matches the published word. Linear stability is NOT SHOWN: the stability test did not finish, and NOT SHOWN is neither a stability nor an instability finding. No Morse index and no KAM statement is made for this orbit.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: NOT SHOWN. Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/published-lists/published-list-orbits.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 27: the orbit that repeats the figure-eight word 17 times, period 116.22, existence

Credit: The interval Krawczyk method of Kapela and Zgliczyński (2003), used here by Sonny. A known orbit, certified here. Tools: Arb (Fredrik Johansson) through python-flint.

For the setup above, an exact rational box about 2×10^{-23} wide in each velocity and 4×10^{-23} in the period, around the orbit that follows the figure-eight word repeated 17 times (period, rounded, 116.220683), contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-23}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.553779) and the least period is the box period. A separate exact test at one third of the period rules out an equal-phase choreography. The numerical multipliers lie far from the unit circle, which indicates that the orbit is linearly unstable; they are computed numbers, so linear stability is NOT SHOWN (multipliers COMPUTED only). The apparent match with the Euler row 260854; 821-satellite row 190 is numerical; a formal identification with a catalogue entry is OPEN.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: NOT SHOWN (multipliers COMPUTED only). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/figure-eight-powers/repeat-17-period-116/orbit.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 28: the orbit that repeats the figure-eight word 18 times, period 85.49, existence

Credit: The interval Krawczyk method of Kapela and Zgliczyński (2003), used here by Sonny. A known orbit, certified here. Tools: Arb (Fredrik Johansson) through python-flint.

For the setup above, an exact rational box about 2×10^{-23} wide in each velocity and 4×10^{-23} in the period, around the orbit that follows the figure-eight word repeated 18 times (period, rounded, 85.493542), contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-23}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.290206) and the least period is the box period. A separate exact test at one third of the period rules out an equal-phase choreography. The numerical multipliers lie far from the unit circle, which indicates that the orbit is linearly unstable; they are computed numbers, so linear stability is NOT SHOWN (multipliers COMPUTED only). The apparent match with the Euler row 295352; 821-satellite row 205 is numerical; a formal identification with a catalogue entry is OPEN.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: NOT SHOWN (multipliers COMPUTED only). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/figure-eight-powers/repeat-18-period-85/orbit.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 29: the orbit that repeats the figure-eight word 11 times, period 55.91, existence

Credit: The interval Krawczyk method of Kapela and Zgliczyński (2003), used here by Sonny. A known orbit, certified here. Tools: Arb (Fredrik Johansson) through python-flint.

For the setup above, an exact rational box about 2×10^{-23} wide in each velocity and 4×10^{-23} in the period, around the orbit that follows the figure-eight word repeated 11 times (period, rounded, 55.913729), contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state

at every shooting node to within 10^{-23}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.285147) and the least period is the box period. A separate exact test at one third of the period rules out an equal-phase choreography. The numerical multipliers lie far from the unit circle, which indicates that the orbit is linearly unstable; they are computed numbers, so linear stability is NOT SHOWN (multipliers COMPUTED only). The apparent match with the Euler row 72771; 821-satellite row 85 is numerical; a formal identification with a catalogue entry is OPEN.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: NOT SHOWN (multipliers COMPUTED only). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/figure-eight-powers/repeat-11-period-56/orbit.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 30: the orbit that repeats the figure-eight word 10 times, period 77.42, existence

Credit: The interval Krawczyk method of Kapela and Zgliczyński (2003), used here by Sonny. A known orbit, certified here. Tools: Arb (Fredrik Johansson) through python-flint.

For the setup above, an exact rational box about 2×10^{-23} wide in each velocity and 4×10^{-23} in the period, around the orbit that follows the figure-eight word repeated 10 times (period, rounded, 77.424620), contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-23}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.234238) and the least period is the box period. The separate exact test at one third of the period that would rule out an equal-phase choreography does not decide: NOT SHOWN. The numerical

multipliers lie far from the unit circle, which indicates that the orbit is linearly unstable; they are computed numbers, so linear stability is NOT SHOWN (multipliers COMPUTED only). The apparent match with the Euler row 49771; 821-satellite row 59 is numerical; a formal identification with a catalogue entry is OPEN.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: NOT SHOWN (multipliers COMPUTED only). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/figure-eight-powers/repeat-10-period-77/orbit.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 31: the orbit that repeats the figure-eight word 9 times, period 74.73, existence

Credit: The interval Krawczyk method of Kapela and Zgliczyński (2003), used here by Sonny. A known orbit, certified here. Tools: Arb (Fredrik Johansson) through python-flint.

For the setup above, an exact rational box about 2×10^{-23} wide in each velocity and 4×10^{-23} in the period, around the orbit that follows the figure-eight word repeated 9 times (period, rounded, 74.734329), contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-23}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.179527) and the least period is the box period. A separate exact test at one third of the period rules out an equal-phase choreography. The numerical multipliers lie far from the unit circle, which indicates that the orbit is linearly unstable; they are computed numbers, so linear stability is NOT SHOWN (multipliers COMPUTED only). The apparent match with the Euler row 30084; none in the 821-satellite file is numerical; a formal identification with a catalogue entry is OPEN.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: NOT SHOWN (multipliers COMPUTED only). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/figure-eight-powers/repeat-9-period-75/orbit.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 32: the orbit that repeats the figure-eight word 10 times, period 72.92, existence

Credit: The interval Krawczyk method of Kapela and Zgliczyński (2003), used here by Sonny. A known orbit, certified here. Tools: Arb (Fredrik Johansson) through python-flint.

For the setup above, an exact rational box about 2×10^{-23} wide in each velocity and 4×10^{-23} in the period, around the orbit that follows the figure-eight word repeated 10 times (period, rounded, 72.916266), contains the starting velocities and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-23}, and in that full box this orbit gives the only solution of the multiple-shooting equations; there are no collisions (pair distance lower bound 0.147755) and the least period is the box period. A separate exact test at one third of the period rules out an equal-phase choreography. The numerical multipliers lie far from the unit circle, which indicates that the orbit is linearly unstable; they are computed numbers, so linear stability is NOT SHOWN (multipliers COMPUTED only). The apparent match with the Euler row 49787; none in the 821-satellite file is numerical; a formal identification with a catalogue entry is OPEN.

Status: existence, absence of collisions and least period: certified (computer-assisted). Linear stability: NOT SHOWN (multipliers COMPUTED only). Morse index: not computed. KAM: not claimed.

Proof: chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits); certificates/figure-eight-powers/repeat-10-period-73/orbit.json; recipes in VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Proposition 33: the thirty orbits are pairwise distinct

Credit: This result (Sonny): certified.

Across the book there are 32 certified boxes. They describe 30 pairwise distinct orbits including the figure-eight, and 29 distinct orbits with certified existence besides the figure-eight: 13 from the 2013 table (15 rows, since rows II.C.2a and II.C.2b are one orbit and rows II.C.3a and II.C.3b are one orbit), 10 from published lists and 6 figure-eight-power orbits.

Status: certified (computer-assisted); the check guide gives the test for distinctness.

Proof: VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify), distinctness check; statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement), "Counts for the whole book".

Proposition 34: yin-yang II is linearly stable at both scales

Credit: Builds on: the reversible stability criterion of Roberts (2007). This result (Sonny): certified.

Rows II.C.3a and II.C.3b of the 2013 table are one orbit seen at two scales. Two separate computations at the two scales give overlapping enclosures of the scale-invariant quantity $T^2|E|^3$ (about 14017.58), and cosines that agree to about 3×10^{-45}. The orbit is linearly stable, with a cosine within 2.31×10^{-9} of 1.

Status: certified (computer-assisted); the stability verdict can be rechecked from the saved final matrices with exact fractions (check 3 of the check guide).

Proof: chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked), section "The same orbit, twice"; transported matrices in certificates/yin-yang-II-3a/ and certificates/yin-yang-II-3b/; VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Proposition 35: a second flow code on the fifteen rows, partial confirmation

Credit: Tools: Arb through python-flint; CAPD. This result (Sonny): certified.

A second flow code, based on Arb and different from CAPD, was run on the same fifteen rows, starting from the middle of each box. It certified an orbit inside the book's box for 9 of the 15 rows, which is 8 distinct orbits: butterfly I, goggles, yin-yang I (both rows), butterfly III, moth I, dragonfly, moth III and moth II. For 8 of those 9 rows its linear verdict is the same as the book's (butterfly I, goggles, yin-yang I twice, dragonfly: unstable; butterfly III, moth I, moth II: stable); for moth III its stability test did not finish. It found no opposing verdict. For the other 6 rows (butterfly II, bumblebee, butterfly IV, yarn, and yin-yang II at both scales) the test did not finish within the time given: NOT SHOWN, not a disproof.

Status: certified (computer-assisted) for the 9 rows; NOT SHOWN for the other 6 rows. The book says this indicates that the nine certified rows do not depend on a fault specific to the CAPD-based code, as far as a second code can show.

Proof: certificates/second-code/second-code-rows.json; chapters/02-what-was-checked.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-02-what-was-checked); VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify).

Theorem 36: invariant two-tori accumulate at moth I

Credit: Builds on: the KAM method of Kapela and Simó (2017). Tools: SAT solvers; Arb through python-flint; CAPD. This result (Sonny): conditional.

Under the classical KAM theorem, invariant two-tori accumulate at the orbit moth I (row I.B.1), and motion starting on them stays on them. The return map, four-dimensional after the conserved quantities are removed, has moth I as a fixed point. It was assembled from eight section-to-section pieces, each computed at 384-bit precision with its derivatives to third order. Four facts about the map at the fixed point were checked: the two rotation frequencies are simple and elliptic; no low-order resonance holds between them (112 small divisors, each separated from zero); the cubic normal form has the required shape (112 coefficients exactly zero); the twist is nondegenerate, the 2×2 determinant D being not zero, with D about -1230.0886 and interval $[-1230.08884821, -1230.08833806]$, half-width 2.6×10^{-4}, strictly negative. The result gives no size for the tori and does not rule out slow drift between them. No other row has a KAM result.

Status: accepted, conditional on four stated premises (check guide): (1) the existence and collision-free certificates for moth I, which come from this book's own existence computation of the fifteen rows, are sound; (2) the flow, derivative, interval-time and event computations of the CAPD/MPFR solver are sound; (3) the build and the solver are correct (a second solver, using Arb through python-flint with no CAPD code, puts the twist determinant inside the stated interval); (4) the classical local KAM theorem for analytic symplectic maps (admitted mathematics, in the form used by Kapela and Simó, 2017). A separate native test of the solver's restart path with interval times, run against exact rational reference values, passed; it supports premise 2 and does not remove it. The certificate files for the finite conditions carry a field KAM_accepted set to false: the programs that wrote them check the finite conditions only and never apply the theorem.

Proof: chapters/03-moth-i.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-03-moth-i); VERIFY.md (https://gocheckyourself.org/read/three-body/experts/#ch-verify), check 4; certificates in certificates/moth-I/; conditional shape in Lean, entries 37 to 43.

Theorem 37: a KAM criterion applied to the seven hypotheses gives invariant tori

Credit: Tools: Lean 4 (Leonardo de Moura, Sebastian Ullrich and the Lean community). This result (Sonny): proved (Lean).

If seven stated hypotheses hold (a fixed point, analyticity, a symplectic map, elliptic spectrum, no low-order resonance, the required normal form, a nondegenerate twist) and a KAM criterion, an implication from the conjunction of the seven hypotheses to the conclusion that tori exist, is supplied as an argument, then the invariant tori follow. In symbols, for propositions $p_1, \ldots, p_7$ and tori: if $\mathrm{Hyp}(p_1, \ldots, p_7) \to \mathrm{tori}$ and $\mathrm{Hyp}(p_1, \ldots, p_7)$, then tori.

```
-- name: MothKAM.applyCriterion
theorem applyCriterion
    {fixedPoint analytic symplectic elliptic nonresonant canonical twist to
    (criterion : Hypotheses fixedPoint analytic symplectic elliptic nonreso
    (h : Hypotheses fixedPoint analytic symplectic elliptic nonresonant can
  criterion h
```

Status: proved and checked in Lean; depends on no axioms. The classical KAM criterion remains an explicit argument, never an axiom. Lean does not check the KAM theorem itself.

Proof: lean/KAMBinding.lean; Lean theorem named in the code block above; chapters/03-moth-i.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-03-moth-i), section "What Lean covers".

Theorem 38: the twist determinant alone does not supply the other six hypotheses

Credit: Tools: Lean 4 (Leonardo de Moura, Sebastian Ullrich and the Lean community). This result (Sonny): proved (Lean).

The seven-hypothesis structure cannot hold when the fixed-point hypothesis is false, even if the other six are true:

Hypotheses(False, True, True, True, True, True, True) → False. The book reads this as: the twist determinant alone does not supply the six other hypotheses.

```
-- name: MothKAM.determinantAloneDoesNotFillTheGate
theorem determinantAloneDoesNotFillTheGate
    (h : Hypotheses False True True True True True True) : False := h.fixed
```

Status: proved and checked in Lean; depends on no axioms.

Proof: lean/KAMBinding.lean; Lean theorem named in the code block above.

Lemma 39: the endpoint denominators of the twist determinant are positive

Credit: Tools: Lean 4 (Leonardo de Moura, Sebastian Ullrich and the Lean community). This result (Sonny): proved (Lean).

Write the exact rational enclosure of the twist determinant as the interval from L_n/L_d to U_n/U_d (the integers are copied into the Lean file). Then $L_d > 0$ and $U_d > 0$.

```
-- name: MothKAM.endpointDenominatorsPositive
theorem endpointDenominatorsPositive : And (lowerDenominator > 0) (upperDen
```

Status: proved and checked in Lean; depends on no axioms. Lean does not check the numbers (they are copied in as integers) or the link between the saved files and those integers.

Proof: lean/KAMBinding.lean; Lean theorem named in the code block above.

Lemma 40: the endpoints of the twist determinant are in order

Credit: Tools: Lean 4 (Leonardo de Moura, Sebastian Ullrich and the Lean community). This result (Sonny): proved (Lean).

With L_n, L_d, U_n, U_d as in the previous entry, $L_n\, U_d \le U_n\, L_d$.

```
-- name: MothKAM.endpointOrder
theorem endpointOrder : lowerNumerator * upperDenominator <= upperNumerato
```

Status: proved and checked in Lean; depends on no axioms.

Proof: lean/KAMBinding.lean; Lean theorem named in the code block above.

Lemma 41: the upper endpoint of the twist determinant is negative

Credit: Tools: Lean 4 (Leonardo de Moura, Sebastian Ullrich and the Lean community). This result (Sonny): proved (Lean).

$U_n < 0$ (with $U_d > 0$ by Lemma 39, so the upper endpoint is negative).

```
-- name: MothKAM.strictNegativeUpper
theorem strictNegativeUpper : upperNumerator < 0 := by decide
```

Status: proved and checked in Lean; depends on no axioms.

Proof: lean/KAMBinding.lean; Lean theorem named in the code block above.

Lemma 42: an outward lower bound on the twist determinant

Credit: Tools: Lean 4 (Leonardo de Moura, Sebastian Ullrich and the Lean community). This result (Sonny): proved (Lean).

In units of $1/1000000$: $-1230088849 \cdot L_d \le L_n \cdot 1000000$.

```
-- name: MothKAM.outwardLower
theorem outwardLower : (-1230088849 : Int) * lowerDenominator <= lowerNume
```

Status: proved and checked in Lean; depends on no axioms.

Proof: lean/KAMBinding.lean; Lean theorem named in the code block above.

Lemma 43: an outward upper bound on the twist determinant

Credit: Tools: Lean 4 (Leonardo de Moura, Sebastian Ullrich and the Lean community). This result (Sonny): proved (Lean).

In units of $1/1000000$: $U_n \cdot 1000000 \le -1230088338 \cdot U_d$.

```
-- name: MothKAM.outwardUpper
theorem outwardUpper : upperNumerator * 1000000 <= (-1230088338 : Int) * up
```

Status: proved and checked in Lean; depends on no axioms.

Proof: lean/KAMBinding.lean; Lean theorem named in the code block above.

Theorem 44: a periodic orbit at masses 1, 1, 3/2 that the catalogues checked do not list

Credit: Builds on: the catalogues it was compared with, credited on the three-body credits page; the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. Tools: mpmath. The catalogues it was compared with are credited in the credits page. This result (Sonny): comparison with the 23 sets of rows computed.

For masses $1, 1, 3/2$, $G = 1$, zero total angular momentum, with the two lighter stars starting 2 apart and the heavier star midway between them, a certified box about 2×10^{-25} wide in each speed contains the starting speeds and period of a periodic orbit. The full certified shooting box, of which this box is the start-value and period part, also holds the state at every shooting node to within 10^{-25}, and in that full box this orbit gives the only solution of the multiple-shooting equations. No two stars collide along it (the closest any two come is more than 0.00764), its least period is about 9.2391238167, and it is linearly unstable: the growth factor of a small nudge over one period lies between -427.013640011164 and -427.013640011163. Compared with 23 sets of catalogue rows, no row has its masses and angular momentum.

Status: certified (computer-assisted): existence, no collisions, least period and uniqueness (of the multiple-shooting solution in the full certified shooting box). Linear instability: proved (computer-assisted), and a separate exact-

fraction replay agrees; a second, separately written interval check of the instability agrees. An independent run of the equations by ordinary numerical integration, written separately, agrees. Comparison with the 23 sets of rows: computed. Still open: Doedel's continuation data near these masses, and lists available on request, were not obtained; completeness of the literature.

Proof: the article Certified orbits that the catalogues do not list; the table in statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement), section "Three orbits at unequal masses". The certificate files of these three orbits are in the check guide, Check 11.

Theorem 45: a linearly stable periodic orbit at masses 1, 1, 31/16

Credit: Builds on: the family, listed by Li, Jing and Liao (2018) at mass 2; the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. This result (Sonny): certified.

For masses $1, 1, 31/16$, $G = 1$, there is a certified periodic orbit with least period 12.6204700390924. The closest two stars come is more than 0.0402, and it is linearly stable (small nudges just wobble). It belongs to a family that Li, Jing and Liao listed in 2018 at mass 2 (their row I.Ai.c.1 (2)); no set we checked lists mass 31/16.

Status: certified (computer-assisted): existence, least period, closest approach and linear stability. An independent run of the equations by ordinary numerical integration, written separately, agrees. Still open: completeness of the literature; the stable stretch along the family, from about 1.76 to 2.10, is computed on a grid, not proved.

Proof: the article Certified orbits that the catalogues do not list; the table in statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement), section "Three orbits at unequal masses". The certificate files of these three orbits are in the check guide, Check 11.

Theorem 46: a linearly stable periodic orbit at masses 1, 1, 97/80, in the family of moth I

Credit: Builds on: the family of moth I (row I.B.1), found by Šuvakov and Dmitrašinović (2013); the interval Krawczyk method of Kapela and Zgliczyński (2003), for existence; the method of Kapela and Simó (2007), for linear stability. This result (Sonny): certified.

For masses $1, 1, 97/80$, $G = 1$, there is a certified periodic orbit with least period 13.4210698226726. The closest two stars come is more than 0.0777, and it is linearly stable. It belongs to the family of the 2013 row I.B.1, moth I, listed at equal masses; no set we checked lists mass 97/80.

Status: certified (computer-assisted): existence, least period, closest approach and linear stability. An independent run of the equations by ordinary numerical integration, written separately, agrees. Still open: completeness of the literature; the two stable stretches along the family (about 0.96 to 1.032, and about 1.194 to 1.23) are computed on a grid, not proved; the KAM result for moth I is at equal masses, and at mass 97/80 it is open.

Proof: the article Certified orbits that the catalogues do not list; the table in statement/STATEMENT.md (https://gocheckyourself.org/read/three-body/experts/#ch-statement-statement), section "Three orbits at unequal masses". The certificate files of these three orbits are in the check guide, Check 11.

Open questions

Morse index and KAM for the sixteen newer orbits

Credit: Sonny.

The Morse index and the KAM question for the sixteen orbits of chapter 4. The three stable orbits from the Hristov list (#5, #9, #10) are natural candidates; whether the conditions hold has not been tested.

Status: open (not computed).

Proof: none; see chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits).

Linear stability of three orbits from the published lists

Credit: Sonny.

The linear stability of the orbits Hristov list #29, #67 and #87: the stability test did not finish.

Status: NOT SHOWN.

Proof: none; see chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits).

Formal identity of the six figure-eight-power orbits with catalogue entries

Credit: Builds on: the catalogue rows of Hristov and Hristova's Euler list and the 821-satellite list.

A formal statement that each certified orbit is the catalogue entry it numerically matches is not available. The book also regards the identification of the orbit with 11 repeats and period 55.91 with an approximate orbit in Šuvakov (2014) as conjectured.

Status: open (the book's label: OPEN; for the Šuvakov (2014) identification, conjectured).

Proof: none; see chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits).

Size of the tori around moth I, and slow drift

Credit: Sonny.

The size of the invariant tori around moth I, and whether slow drift between them can occur, are not covered by the result and stay open.

Status: open.

Proof: none; see chapters/03-moth-i.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-03-moth-i).

A search of the rigorous-numerics literature for the sixteen

Credit: Sonny.

The end-to-end checker certifies all sixteen inside the book's boxes. A systematic search of the rigorous-numerics literature for earlier certificates of the sixteen newer orbits has not been done; further research is recommended.

Status: open.

Proof: none; see chapters/04-sixteen-more-orbits.md (https://gocheckyourself.org/read/three-body/experts/#ch-chapters-04-sixteen-more-orbits).

9 An ode to checking

An ode to checking

In 1885 a king paid for a question.
For his sixtieth birthday, Oscar II of Sweden offered
a gold medal and 2,500 kronor
for progress on bodies that pull on one another under gravity.

Henri Poincaré answered,
and on 21 January 1889 the prize was his.

Then the memoir went to press in Stockholm.
Edvard Phragmén, reading the proofs, kept asking about one delicate point.
In late 1889 Poincaré found the error behind it.
He stopped the presses,
paid more than 3,500 kronor for the printing, about a thousand more than
the prize,

and wrote it again.
The corrected memoir, printed in 1890, is where the study of chaos took root.

Defend the kronor:
check the work before it goes out.
A reader who finds a fault is part of the work.

In 2013 Milovan Šuvakov and Veljko Dmitrašinović searched by computer
and found periodic orbits for three equal masses:
fifteen starting points, thirteen orbits, one table.
You have no way to look for something
if you do not know it exists.
They found them, and every certificate of the fifteen rows is built on what
they found.

Each of those fifteen rows now has a certificate.
Interval arithmetic brackets every number,
so the work can be checked again:
that the orbit is there, how it bends, whether it is linearly stable,
and, for moth I, the KAM tori around it, under four stated premises.

We encourage others to take part in checking this work.
We make no assertions outside the stated conditions,
and invite you to join the long history of people checking each other's work.

I keep the certificates. Go check yourself.

We stand on the shoulders of giants.

Dates and sums are from the Institut Mittag-Leffler's history of the prize
competition, which cites June Barrow-Green, "Oscar II's prize competition
and the error in Poincaré's memoir on the three body problem", Arch. Hist.
Exact Sci. 48 (1994) 107–131, and Poincaré and the Three-Body Problem

(AMS/LMS, 1997). The sums and dates have not yet been checked against Barrow-Green's book itself.

Where to go next

- What exactly was checked for each of the fifteen rows? See What was checked.
- What is the story behind Poincaré's error? See the kids' version and the credits page.

10 Check it yourself: fifteen periodic three-body orbits, and sixteen more

Check it yourself: fifteen periodic three-body orbits, and sixteen more

This guide gives a written check for each claim in the book: the file format, the condition to test, the answer you should get, and a standard public tool or a few lines of code you can write yourself. Existence, the Morse indices and the other stability verdicts rest on CAPD-based computations whose full files are too large to ship; their results are stated here and are not recomputed by these checks. The Morse acceptance files are withheld for another reason, in the certificate summary's words: The acceptance files carry internal file locations, so they are not shipped; they are available on request, and their fingerprints are given here so that a copy can be matched exactly. This project's own programs are withheld for a different reason, in the check guide's words: How I direct this work, and my own tools, are not part of this book. It uses no program of ours; Python's fractions module and a SHA-256 tool are enough. Each recipe also comes as a step-by-step page with commands for Windows, macOS and Linux and the exact output to expect.

What's in this book

Counts are computed from the book's orbit statement file, its file lists and its Lean file; none is estimated.

Status	Count
Proved and checked in Lean	1 file (Sonny, 2026), 7 theorems in 12 declarations, the 7 theorems depending on no axioms: the conditional KAM implication and exact endpoint arithmetic for moth I; no orbit result is Lean-proved
Written proofs	0
Computed or certified (computer-assisted)	existence for 29 distinct orbits (13 from the 2013 table with 15 rows, 10 from published lists, 6 figure-eight-power orbits), plus the figure-eight as a known case; Morse index certified for 13 orbits (15 rows); linear stability for 20 orbits (9 stable, 11 unstable); for the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists.
Conditional	KAM for moth I, on four premises
Open	Morse index and KAM for the sixteen newer orbits; formal identity of the six figure-eight-power orbits with catalogue entries; a search of the rigorous-numerics literature for prior certificates

Item	Count
Linear stability NOT SHOWN	3 (Hristov lists #29, #67, #87)
Linear stability NOT SHOWN, multipliers COMPUTED, not certified	6 (the figure-eight-power orbits)
Partial second computation of the 2013 rows	9 of 15 rows (8 orbits) certified, 8 linear verdicts agree, none opposes, 6 rows NOT SHOWN
Chapters	4
Certificate files	186 (including the fingerprint list, the Morse-index acceptance summary and the list of renamed or edited certificate files)
Written check recipes	11, including the fingerprint check
Lean files	1
Files in this folder	236

What is claimed

The question these claims answer: for each row of the 2013 table of periodic orbits of three equal masses, is the orbit really there, how does it bend, is it linearly stable, and, for moth I, do invariant tori surround it?

The fifteen rows of Table I in Šuvakov and Dmitrašinović (2013) describe thirteen distinct periodic orbits of three equal masses (rows II.C.3a and II.C.3b are one orbit at two scales; so are II.C.2a and II.C.2b). For each row:

Claim	Result	Status
Existence: a true periodic orbit lies in a small exact box near the reported start, free of collisions, with the stated minimal period and the 2013 shape word	15 of 15	Certified (computer-assisted)
Morse index (fixed period, centre of mass fixed, counted in two symmetry sectors, plus and minus)	15 of 15 certified; in the check guide's words, existence, the Morse indices and the other stability verdicts are stated here and are not recomputed by these checks	Certified (computer-assisted) for all 15 rows; for butterfly IV one block needed an extra exact step (see Morse-index evidence)
Linear stability (planar, fixed energy, zero angular momentum, modulo time shift and rotation)	15 of 15: 7 stable, 8 unstable	Certified (computer-assisted)
KAM stability of moth I: invariant two-tori accumulate at the orbit	moth I only	Accepted, conditional on four premises (below)

The per-row values are in the statement files. The figure-eight is earlier published work and is not a claim here: its existence was proved by Chenciner and Montgomery (2000), after Moore's numerical discovery (1993); its rigorous linear stability is by Kapela and Simó (2007), and its KAM stability by Kapela and Simó (2017). It is run through the same tests as a known case.

Sixteen more orbits (chapter 4). These are orbits that other people had found by numerical search and listed; what is claimed is our own certificate for each.

Claim	Result	Status
Existence of a periodic orbit in an exact box (about 2×10^{-25} wide in each velocity for the ten orbits from published lists; about 2×10^{-23} for the six figure-eight-power orbits), collision-free, with the stated least period and, for the ten, the published crossing word	16 of 16	Certified (computer-assisted)
Linear stability of the ten orbits from published lists	7 of 10: 3 stable, 4 unstable; 3 NOT SHOWN	Certified (computer-assisted) for the 7; NOT SHOWN for Hristov lists #29, #67, #87
Linear stability of the six figure-eight-power orbits	NOT SHOWN; multipliers COMPUTED only (largest moduli about 4 to 87,000)	The computed numbers indicate instability; not certified; further research is recommended
Equal-phase choreography excluded at one third of the period (six figure-eight-power orbits)	5 of 6; the repeat-10 orbit with period 77.4246 is NOT SHOWN	Certified (computer-assisted) for the 5
The 30 orbits (13 in the 2013 table, 10 and 6 here, and the figure-eight) are pairwise different	30 distinct orbits in 32 boxes	Certified by disjoint exact intervals of the quantity T^2 E 3
Apparent match of the six with published catalogue rows	6 of 6	Computed, numerical; formal identity OPEN
Morse index, KAM	none for the sixteen	Not computed

A second computation with different software (Arb in place of CAPD) was run on the same 15 rows, starting from the middle of each box. It certified an orbit inside the book's box for 9 of the 15 rows (8 distinct orbits, since II.C.2a and II.C.2b are one orbit), and its linear verdict agrees with the book's for 8 of them (for moth III the stability test did not finish). It found no opposing verdict. For the other 6 rows (butterfly II, bumblebee, butterfly IV, yarn, and yin-yang II at both scales) the test did not finish, which is NOT SHOWN and is not a disproof. This is a partial confirmation. The row table is in the second-computation file (see the file list in the last section).

Morse-index evidence. Two summary files are shipped. The acceptance summary covers all fifteen rows and the figure-eight. For each it gives the plus and minus values, the route of the computation in plain words, and the

SHA-256 fingerprint and size of the acceptance file that the computation wrote. All fifteen rows were accepted by one of three interval routes (with one added exact step for butterfly IV): a comparison of the second variation with lower and upper operators followed by exact interval inertia (nine rows), a direct interval computation along the orbit (three rows), or a high-precision residual check of the saved interval form (the two yin-yang II rows and butterfly IV). For butterfly IV, one 2 x 2 block in the minus sector was too close to singular for the standard inertia test and was decided by an exact pivot-sign step (every pivot strictly signed, so the count holds for every symmetric matrix in the block); with it a different preconditioner was used for the interval elimination. Both changes run only where the standard test refused, both were reviewed line by line, and the computation was rerun in a pinned Linux setting with the original library builds, reproducing $121 = 60 + 61$ and both residual bounds bit for bit. Two of the 147 orbit segments behind butterfly IV did not finish on the original machine within its time cap; they were computed locally with the same program, with only the time cap changed, and two separate local runs agree byte for byte. Later, the changed inertia code (with the exact pivot-sign step and the exact preconditioner seed) was run on the three known-answer controls, figure-eight (index 2), moth I (index 9) and yin-yang II 3a (index 53); all three matched, with the same residual bounds as the earlier runs of the unchanged code (control runs available on request). The extra exact step was not needed on any control block, so those runs show that the changed code still gives the known answers; they do not exercise the extra step itself. The minus sector of butterfly IV still rests on that one near-singular block. The acceptance files (about 40 MB in all) carry internal file locations and are available on request; their fingerprints are in the summary. The witness summary covers the finite arithmetic witnesses behind five of the indices: figure-eight ($1 + 1$), moth I ($4 + 5$), yin-yang II at both scales ($26 + 27$ each) and butterfly IV ($60 + 61$). For each sector it gives the residual bound (below 1 in every case) and the number of negative directions found by an exact elimination of the saved interval matrix. The full witnesses (about 405 MB for the small orbits, 34 GB for yin-yang II and 6.4 GB for butterfly IV) are not shipped for size and are available on request. Limits: every route shares one ODE code and one inertia

code, so they are not independent of each other; no separately written end-to-end checker from the orbit to the saved matrix exists; and for moth I an additional direct interval check of the saved form did not finish (it is neither a confirmation nor a refutation).

Status

- Existence and linear stability: certified (computer-assisted), 15 of 15.
- Morse index: certified (computer-assisted) for all 15 rows. For the Morse indices, in the certificate summary's words: No separately written end-to-end checker from the orbit to the saved matrix exists. For butterfly IV ($121 = 60 + 61$) one near-singular block in the minus sector was decided by an extra exact step; see Morse-index evidence above.
- Moth I KAM: accepted, conditional on four stated premises:
    1. the existence and collision-free certificates for moth I, which come from this book's own existence computation of the fifteen rows, are sound;
    2. the flow, derivative, interval-time and event computations of the CAPD/MPFR solver are sound;
    3. the build and the solver are correct (a second solver, using Arb through python-flint with no CAPD code, puts the twist determinant inside the stated interval);
    4. the classical local KAM theorem for analytic symplectic maps (admitted mathematics, in the form used by Kapela and Simó, 2017).
- Support for premise 2: a separate native test of the solver's restart path with interval times, run against exact rational reference values, passed. It supports premise 2 and does not remove it.

- No other row has a KAM result. The unstable rows cannot have one.
- The sixteen orbits of chapter 4: existence, collision exclusion and least period certified for all sixteen; linear stability certified for 7 of the ten orbits from published lists. The certificates are our own computation. The published lists pointed the way to these orbits; their data files are not redistributed here, they carry no stated licence, and rights in them

remain with their authors. Anyone who wants to compare with the published rows should take the rows from the authors. The rounded published periods quoted in the ten-orbit file are locators for the list entries, not copies of the files.

- Lean. One short Lean 4 file (toolchain pinned beside it; no outside libraries) proves the conditional implication "if the seven stated hypotheses hold and a KAM criterion is supplied as an argument, then invariant tori follow", that the twist determinant alone does not supply the six other hypotheses, and some exact endpoint arithmetic on moth I's twist determinant (positive denominators, endpoint order, a negative upper endpoint, two outward bounds). Lean does not check the KAM theorem itself, the numbers (they are copied in as integers), the link between saved files and those integers, or the ODE enclosures.
- Reading status: this work has not yet been read by anyone outside this project.

Commands

Everything below uses standard tools: a SHA-256 program, Lean, and a few lines of code in any language with exact rational arithmetic (Python's fractions module is enough). Run from the book folder.

```
sha256sum -c certificates/SHA256SUMS.txt          # Linux
shasum -a 256 -c certificates/SHA256SUMS.txt   # macOS
CertUtil -hashfile <file> SHA256                  # Windows, one file at a
cd lean
lake build                                        # needs the Lean version
```

Number formats used in the certificate files

- An exact fraction is written as a string such as "-12345/6789" or as an integer string. An interval is a pair [lo, hi] with lo $\leq$ hi.
- A dyadic number is a pair [m, e] of an integer string and an integer, meaning m $\times$ 2^e. A dyadic interval is a pair of dyadic numbers. In the

moth I files a dyadic endpoint is written {"m": "...", "e": ...} with the same meaning.

- Interval arithmetic is exact: $[a, b] + [c, d] = [a + c, b + d]$; $[a, b] - [c, d] = [a - d, b - c]$; a product is the minimum and maximum of the four endpoint products. A strict claim such as "inside $(-1, 1)$" means the whole interval is inside.

Check 1. The files are the ones we published

Compute the SHA-256 of every file listed in certificates/SHA256SUMS.txt (each line is a hash, a space, a star and a path relative to the book folder). Every value must match exactly. Expected: all listed files match.

Check 2. Existence boxes and the 2013 table

Files: the existence boxes file (16 entries, the figure-eight and the 15 rows) and the orbit statement file (key "table_2013" holds the published vx, vy and T as decimal strings).

For each entry: "velocity_box_exact" has intervals vx and vy; "period_box_exact" and half_period_box_exact are intervals. Test that the period interval equals twice the half-period interval, that collision_free, minimal_period_checked and shape_word_matched are true, and that every interval is narrower than 10^{-21}. For each table row, the midpoint of the vx box and of the vy box must be within one unit of the last printed digit of the published value. Also compute, with the exact interval arithmetic above, the quantity

$$Q = T^2 \cdot (5/2 - 3(vx^2 + vy^2))^3$$

(that is $T^2|E|^3$ for energy $E = -5/2 + 3(vx^2 + vy^2)$). The two Q intervals for yin-yang II rows II.C.3a and II.C.3b must overlap, at about 14017.5797.

Expected: all 16 boxes well formed; velocities of all 15 rows agree with the 2013 table; the certified periods differ from the 2013 periods by between 1.467×10^{-4} and 2.0×10^{-3} (butterfly I -9.25×10^{-4}, moth I $+4.05 \times 10^{-4}$, butterfly IV -9.46×10^{-4}, yin-yang II rows -4.73×10^{-4} and $+3.98 \times 10^{-4}$).

Check 3. Linear stability of yin-yang II, from the saved final matrices

Files: for each of the two scales, a "reversible" file and a "quotient" file. No differential equation is integrated.

Route 1. The reversible file holds cosine_matrix_exact, a 2×2 matrix of intervals. With $t = c00 + c11$ and $d = c00 \cdot c11 - c01 \cdot c10$ (interval arithmetic) test: $t^2 - 4d > 0$, $1 - t + d > 0$, $1 + t + d > 0$, and $-2 < t < 2$ on the whole interval. These say that the quadratic $c^2 - tc + d$ has two real, distinct roots strictly inside $(-1, 1)$. The stored cosine_eigenvalues_exact must be two disjoint intervals inside $(-1, 1)$.

Route 2. The quotient file holds full_quotient_exact, a 4×4 matrix M of intervals. Let $a =$ trace M, $b =$ the sum over $i < j$ of (M_ii·M_jj − M_ij·M_ji), and $c = b - 2$. Test: $a^2 - 4c > 0$, $4 - 2a + c > 0$, $4 + 2a + c > 0$, and $-4 < a < 4$. The stored pair_trace_parameters_exact, each halved, must each meet one of the stored cosine eigenvalue intervals.

Expected at both scales: both routes pass and agree; the two cosines are about -0.0322 and $1 - 2.30 \times 10^{-9}$ (within 2.31×10^{-9} of 1, the figure used elsewhere in this book). That is, the orbit is linearly stable, though only barely.

Check 4. Moth I: the finite KAM conditions

Files: four files in the final torsion folder (the one with eight section nodes): the node list, the determinant and eigenvalue file, the divisor file and the normal-form file. In these files each endpoint is {"m", "e"}.

- Nodes: the selected nodes are 0 to 7 and no fallback entry is present.
- Twist: the real part of complex_determinant is an interval entirely below zero. Expected: $[-1230.08884821, -1230.08833806]$.
- No low-order resonance: the divisor file lists divisors, each with a component i (0 to 3), a multi-index alpha of four non-negative integers, and a complex divisor (real and imaginary intervals). Call a divisor structural if sum(alpha) $= 3$ and, for some j in $\{0, 1\}$, alpha_k $= [k = i] + [k = j] + [k = j + 2]$ for $k = 0$ to 3 (zero-based positions). For every

non-structural divisor the modulus must exceed 10^{-10}, taking the smallest possible modulus over the box. Expected: 112 of 112 non-structural divisors separated from zero; 8 structural divisors, which are allowed to contain zero.

- Cubic normal form: for every component i and every alpha of total degree 2 or 3 that is not structural and is present in the normal-form file, the real and imaginary intervals of the coefficient must both be exactly $[0, 0]$. Expected: 112 of 112.
- Ellipticity: the four stored eigenvalues each have a real-part interval strictly inside $(-1, 1)$.

Expected: all conditions hold. The eigenvalues of the return map are about 0.5410529 ± 0.8409886i and -0.9986992 ± 0.0509891i. An earlier set of the same type (one section node only, with broad derivative enclosures) is also in the folder as an earlier result. Its twist interval is $[-2.0 \times 10^{118}, 2.0 \times 10^{118}]$ and contains zero, so it does not meet the conditions and is not used for any result.

The determinant and normal-form files of both sets carry a field named KAM_accepted with the value false. That is expected and is not a refusal: the programs that wrote these files check the finite conditions only and never apply the KAM theorem, so they never set that field to true. The conditional acceptance is the separate step of applying the theorem to the conditions checked here, under the four premises in the Status section.

Check 5. Ten orbits from published lists

File: the ten-orbit file, with ten orbits. Each has box (intervals vx, vy, T as exact fractions), "collision_distance_lower", and gates.

- Each box is narrower than 10^{-24} in every coordinate, the quantity Q computed as in Check 2 is defined (energy negative), and "collision_distance_lower" is positive.
- The stored gates are all true: strict_krawczyk_inclusion, minimal_period_excluded and shape_word_matches_published;

contraction_norm_upper_approx < 1; max_K_over_R_approx < 1; collision_clearance_lower_approx > 0.

- Linear verdict, for the seven orbits that have a derivative-factor file. The factors file holds 64 matrices of size 8×8 (51 for Li–Liao I.A_3; dyadic intervals), the factors of the flow from section to section. Form the monodromy A as follows. If the file has "basis" (one point matrix p_i for each factor), set $P = I$ and $B = I$ and for each factor f_i in order compute $H = I + p_i^{-1}\,(f_i\,P - p_i)$ (exact inverse, interval arithmetic), set $B = H\,B$ and $P = p_i$, and finally $A = P\,B$. If there is no basis, $A = f_n \cdots f_2\,f_1$, where n is the number of factors. Then let K be the 4×4 matrix with rows (2,0,1,0), (0,2,0,1), (1,0,2,0), (0,1,0,2), O the 8×8 block matrix [[0, K], [−K, 0]], and Oi the block matrix $[[0, -K^{-1}], [K^{-1}, 0]]$. Let S be the 8×8 matrix with $S[i][(i+2) \bmod 4] = -1$ and $S[i+4][4+(i+2) \bmod 4] = +1$ for $i = 0$ to 3. Set $M = S \cdot (Oi \cdot A^{T} \cdot O) \cdot S \cdot A$, $a = \mathrm{trace}(M) - 4$, $b = \tfrac{1}{2}(a^2 - \mathrm{trace}(M^2) + 4)$, and $\Delta = a^2 - 4(b - 2)$. If $\Delta > 0$ the two cosine roots are $(a \pm \sqrt{\Delta})/4$. The orbit is stable if the roots are disjoint and both strictly inside $(-1, 1)$; unstable if one lies entirely outside $[-1, 1]$. Rounding every product outward to a grid of 2^{-384} keeps the arithmetic rigorous.

Expected: 3 stable (Hristov lists #5, #9, #10), 4 unstable (the four Li–Liao orbits), 3 NOT SHOWN (Hristov lists #29, #67, #87, which have no factors file). The cosine roots are about: Hristov #5 0.999964 and 0.477474; Li–Liao I.B_7 1.380195 and −0.263404; Li–Liao I.A_3 42.448493 and 0.080117; Hristov #9 −0.185911 and −0.248670; Li–Liao II.C_2 29.6485 and −99.8885; Hristov #10 0.983494 and 0.408735; Li–Liao II.C_61 472.8949 and 9.734249.

Check 6. The strict Krawczyk inclusion, for two orbits

Folders: one for the Li–Liao I.A_3 orbit (dimension 403) and one for the orbit that follows the figure-eight word 11 times (dimension 507). Each holds a certificate file (dimension n, a radius R, the residual F as n intervals, the derivative enclosure DG as a list of [row, column, interval] entries, and a list of gzip-compressed preconditioner blocks with their SHA-256 hashes and

sizes) and the blocks themselves (JSON with "start" and "rows"; rows are lists of dyadic numbers).

Verify each block's size and SHA-256, assemble the n × n matrix C, then in exact arithmetic compute, for each row i, the value $|(C\cdot F)_i| / R + \Sigma_j |I - C\cdot DG|_{ij}$ (the largest absolute value over the interval entries of each term), and the infinity norm of $I - C\cdot DG$. Strict inclusion holds if the largest row value is below 1 and the norm is below 1. That gives exactly one zero of the shooting equations in the box. The check takes F and DG as given; whether they enclose the differential equation is outside its scope.

Expected: strict inclusion true. For dimension 403 the contraction norm is about 1.28×10^{-18}; for dimension 507 about 2.02×10^{-15}.

Check 7. Six figure-eight-power orbits

Each orbit folder holds an orbit file (dyadic intervals vx, vy, T, E, Q, "clearance_lower"; a syzygy string; figure_eight_word_power k; "least_period") and a one-third-period file.

- The box is narrower than 10^{-22} in vx, vy and T. The Q interval computed from the box as in Check 2 lies inside the stored Q interval, and the stored E contains $-5/2 + 3(vx^2 + vy^2)$. "clearance_lower" is positive and "least_period" is true.
- The syzygy string is "312" repeated 2k times (length 6k), with k the repeat count.
- Q is disjoint from k^2 times the Q of the figure-eight, so the orbit is not the figure-eight run k times.

- The one-third-period file gives flow_endpoint (eight dyadic intervals: x and y of bodies 1 and 2, then their velocities; body 3 follows from the centre of mass), the initial velocity intervals, and two flags that must be true (strict_taylor_interior, full_terminal_interval_covered). Build the state of all three bodies at T/3 and compare it, component by component with interval subtraction, with the start state after each of the two cyclic relabellings of the bodies (1→2→0 and 2→0→1; the start state has positions (−1,0), (1,0), (0,0) and velocities (vx,vy), (vx,vy),

$(-2vx,-2vy)$). If, for both relabellings, at least one of the 12 differences excludes zero, an equal-phase choreography is excluded.

Expected: all six boxes pass; the equal-phase choreography is excluded for 5 of 6 and not decided (NOT SHOWN) for the orbit that repeats the figure-eight word 10 times with period 77.4246. The repeat counts and periods are: 17 (116.2207), 18 (85.4935), 11 (55.9137), 10 (77.4246), 9 (74.7343), 10 (72.9163).

Check 8. The 30 orbits are pairwise different

Compute Q for all 32 boxes: the 16 existence boxes, the 10 boxes of the ten-orbit file and the 6 figure-eight-power boxes. Two orbits that are the same up to scale, rotation, time shift, relabelling or reversal have the same Q, so disjoint Q intervals show that the orbits differ. Expected: exactly two pairs of Q intervals overlap, namely rows II.C.2a and II.C.2b, and rows II.C.3a and II.C.3b, which are declared to be one orbit at two scales each. Hence 32 boxes are 30 distinct orbits.

Check 9. Morse-index summaries

The witness summary file lists five orbits. For each, the two sector counts of negative directions must add up to the total index, must equal the plus and minus values in the orbit statement file, and every stored bound must be below 1. Expected: figure-eight $1 + 1$, moth I $4 + 5$, yin-yang II (both scales) $26 + 27$, butterfly IV $60 + 61$.

The acceptance summary file lists sixteen entries, the figure-eight and the fifteen rows. For each, plus and minus must add up to the total and must equal the values in the orbit statement file, and every SHA-256 must be 64 hexadecimal digits. Expected: 16 of 16 consistent and all 16 entries accepted.

Both checks test consistency only. They do not recheck a witness or an acceptance file.

Check 10. The Lean file

Run lake build in the lean folder. Expected: the build completes, printing seven lines of the form below, one for each of the seven theorems (the file has 7 theorems in 12 declarations), each saying that it does not depend on any axioms.

```
info: KAMBinding.lean:41:0: 'MothKAM.applyCriterion' does not depend on any
info: KAMBinding.lean:42:0: 'MothKAM.determinantAloneDoesNotFillTheGate' do
info: KAMBinding.lean:43:0: 'MothKAM.endpointDenominatorsPositive' does not
info: KAMBinding.lean:44:0: 'MothKAM.endpointOrder' does not depend on any
info: KAMBinding.lean:45:0: 'MothKAM.strictNegativeUpper' does not depend o
info: KAMBinding.lean:46:0: 'MothKAM.outwardLower' does not depend on any a
info: KAMBinding.lean:47:0: 'MothKAM.outwardUpper' does not depend on any a
Build completed successfully (3 jobs).
```

Check 11. Three orbits at unequal masses

Folders: certificates/unequal-masses/masses-1-1-3-2, masses-1-1-31-16 and masses-1-1-97-80. Setup: masses 1, 1 and m3 (3/2, 31/16 or 97/80), G = 1, zero angular momentum; q1 = (−1, 0), q2 = (1, 0), q3 = (0, 0); v1 = v2 = (vx, vy), v3 = −(2/m3)(vx, vy). Each folder holds:

- certificate/krawczyk-bound.json and its preconditioner blocks: the strict Krawczyk inclusion for the multiple-shooting equations, in the format of Check 6 (dimension n, radius R, residual F, derivative enclosure DG, block list with SHA-256 and sizes).
- shooting-nodes.json: the centre of the certified box: vx, vy, the half-period tau and the shooting nodes (time t; state q1x, q1y, q2x, q2y, v1x, v1y, v2x, v2y). Every coordinate of the box is this centre plus or minus R.
- stability.json: the saved interval enclosures of the half-period map A, the reversor R, the form Omega and the full-period matrix M = R Omega^{-1} A^{T} Omega R A (8×8, dyadic intervals).
- orbit.json: the setup, the box centre and radius, the stored lower bound on the closest approach of two bodies (clearance_lower) and the syzygy word.

Tests:

1. Fingerprints: every file matches certificates/SHA256SUMS.txt (as in Check 1).
2. Existence: run Check 6 on certificate/krawczyk-bound.json. Strict inclusion must hold. As in Check 6, F and DG are taken as given.
3. Period: the least period lies in 2(tau − R, tau + R), computed exactly from shooting-nodes.json and the certificate's R.
4. Linear stability, from M in stability.json (no differential equation is integrated): with exact interval arithmetic let $S = \operatorname{trace} M - 4$ and $D = 2\operatorname{trace}(M^2) - S^2$. The four multipliers other than the four neutral ones (energy and rotation) form the reciprocal pairs with sums $u = (S \pm \sqrt{D})/2$. Both pairs lie on the unit circle (stable) if $D > 0$ and both u lie strictly inside $(-2, 2)$. Some multiplier has modulus above 1 (unstable) if $\operatorname{trace} M < -8$, because a real 8×8 matrix whose multipliers all have modulus at most 1 has $|\operatorname{trace}| \le 8$.
5. clearance_lower is positive (taken as stored, as in Check 5).

Expected:

Orbit	**n**	**Contraction norm (about)**	**Period**	**trace M (about)**	**u (about)**	**Verdict**
masses 1, 1, 3/2	451	2.857×10^{-12}	9.23912381674221315289…	−420.88593420278	−427.01598, 2.13005	unstable (trace below −8)
masses 1, 1, 31/16	379	8.680×10^{-15}	12.62047003909235321770…	0.9123448	−1.67780, −1.40985	stable
masses 1, 1, 97/80	307	2.348×10^{-16}	13.42106982267256288734…	5.5242743	0.130025, 1.394249	stable

The period intervals are 2×10^{-25} wide. The stored closest-approach lower bounds are 0.0076462 (3/2), 0.0402700 (31/16) and 0.0777375 (97/80). For the 3/2 orbit the multiplier with u near −427.016 is about −427.01364, the growth factor of Theorem 44.

Other checks of these orbits (not part of this recipe):

- An independent ordinary numerical integration of the full equations (all three bodies explicit), written separately, from the box centre: a Taylor series method at 40 significant digits returns to the start after one certified period to within 7.3×10^{-37} (3/2) and 2.4×10^{-38} (97/80); an explicit Runge–Kutta 8(5,3) method at tolerances 10^{-13} and 10^{-14} finds the return times 9.2391238167331, 12.620470039089 and 13.421069822669 (within 1.1×10^{-11} of the certified periods) and closest approaches 0.0076990, 0.0408045 and 0.0792388, each above the certified lower bound. These are COMPUTED.

- A second interval check of the instability of the 3/2 orbit, written separately: its own validated Taylor integration of the flow and its derivative at 100 bits (outward-rounded arithmetic on mpmath's binary floats, in place of Arb at 384 bits), over the 57 segments of the certified box, then an exact-fraction product of the 57 segment derivatives with a rigorous error bound. It gives trace M in $[-420.88611, -420.88576]$, below -8, so a multiplier has modulus above 1; this interval contains the value above. Proved (computer-assisted).

Time and memory: tests 2 to 5 take about two to four minutes per orbit in plain Python with exact fractions, under 100 MB.

Expected output

In short, using the sections above: the fingerprints all match (Check 1); 16 boxes are well formed and 15 rows agree with the 2013 table (Check 2); yin-yang II is stable at both scales (Check 3); the finite KAM conditions hold for moth I with twist determinant in $[-1230.08884821, -1230.08833806]$ (Check 4); the ten-orbit file gives 3 stable, 4 unstable and 3 NOT SHOWN (Check 5); the two Krawczyk inclusions are strict (Check 6); the six figure-eight-power boxes pass and 5 of 6 exclude the equal-phase choreography (Check 7); 30 pairwise distinct orbits in 32 boxes (Check 8); both Morse summaries are consistent (Check 9); the Lean build prints seven lines "does not depend on any axioms" (Check 10); the three unequal-mass orbits pass, with verdicts unstable (3/2), stable (31/16) and stable (97/80) (Check 11).

Time and memory

With exact fractions in a compiled or interpreted language, every check above except the linear verdicts of the ten-orbit file takes seconds and under 100 MB. The linear verdicts of the ten-orbit file (Check 5) multiply 8×8 matrices of very long numbers and take about half a minute with under 500 MB in Python. The two Krawczyk checks take a few seconds each. The Lean build takes a few seconds.

What is not claimed

- These are periodic orbits, not choreographies. Only the specific orbits at the reported starts are covered, not whole families.
- No KAM result for any orbit except moth I, and that one is conditional on the four premises. The KAM conclusion is that invariant tori accumulate at the orbit; no size of torus and no stronger confinement is claimed.
- The checks above recheck the final exact arithmetic (stability of yin-yang II, the finite KAM conditions for moth I, the table comparison, the distinctness of the orbits). Existence, the Morse indices and the other stability verdicts rest on CAPD-based computations whose full files are too large to ship; their results are stated here and are not recomputed by these checks. The Morse acceptance files are withheld for another reason, in the certificate summary's words: The acceptance files carry internal file locations, so they are not shipped; they are available on request, and their fingerprints are given here so that a copy can be matched exactly. This project's own programs are withheld for a different reason, in the check guide's words: How I direct this work, and my own tools, are not part of this book.
- Every row shares one CAPD-based ODE code, so an error in that code would touch every row. The second solver covers moth I's KAM inputs, and the second computation covers 9 of the 15 rows.

- Morse indices depend on the function space. These are fixed-period indices in two symmetry sectors and cannot be compared directly with

indices in other spaces (for example the choreography space used for the figure-eight by Fukuda, Fujiwara and Ozaki).

- Two stable orbits are only barely stable: a reduced cosine within 2.31×10^{-9} of 1 (yin-yang II) and within 3.8×10^{-23} of 1 (butterfly IV). The verdicts hold at the computed widths.

- For the sixteen orbits of chapter 4: no Morse index and no KAM statement. The end-to-end checker certifies all sixteen inside the book's boxes. The checks recheck the boxes, the exact invariants, the distinctness, the seven linear verdicts from saved derivative factors and, for two orbits, the strict Krawczyk inclusion; they take the flow enclosures, and for the other fourteen orbits the existence gates (Krawczyk inclusion, least period, crossing word), as stored. The large matrices of the existence test (12 to 73 MB per orbit, about 436 MB for the sixteen) are shipped for two orbits only, as worked examples (Li–Liao I.A_3 and the orbit that repeats the figure-eight word 11 times); for the other fourteen they are available on request.

- The six figure-eight-power orbits are apparently the same as entries in published catalogues. That is a numerical match to the digits printed there, not a proof of identity, and we did not obtain the earliest publication date of each entry. We did not search the interval-arithmetic literature for earlier certificates of the sixteen; the catalogues and papers we compared are numerical.
- The second computation is partial (9 of 15 rows). Its conversion of the free-group word to a crossing word has not been reviewed, so its shape-word step is not claimed.
- Lean covers only the conditional implication and exact endpoint arithmetic described above. It does not cover any orbit result. The Lean file was checked by a build and its axiom report; no second kernel was run on it.
- The only claim is that the certificates support the stated values. Anyone can recheck the parts shipped here.

How I direct this work, and my own tools, are not part of this book. The files listed above ship; the files named there as not shipped are available on request. In this book the Lean part is one short file: it proves the conditional KAM implication and exact endpoint arithmetic for moth I, and no orbit result.

Supporting records are available on request.

Where to go next

- Which of the four premises behind the moth I result is the hardest to examine, and what would it take to remove it? The question is taken up in Moth I.
- How are the sixteen further orbits different from the fifteen rows, and what is still missing for them? See Sixteen more orbits.

We hope it helps others find more results and keeps their curiosity going.
We'd love to collaborate with anyone.

11 Files and fingerprints

This chapter is the book's file SHA256SUMS.txt.

12 Trust and verification

This chapter is the book's file VERIFY.md.

13 Lean sources: how to check them

This chapter is the book's file THEOREMS.md.

14 People and works

This chapter is the book's file CREDITS.md.

15 Files and rights

Licence

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What it covers: the chapters, the articles, the statement pages, the certificate files, the exact boxes, the fingerprint lists and the short Lean file.

What it does not cover:

- CAPD, which is under the GPL. If any CAPD source or a binary built from it ships, GPL terms apply to it.
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Where to go next

- What does this book show, and how can you check it? See the book's front page and the check guide.

16 Articles

- Certified orbits that the catalogues do not list

17 Each result on its own page

1. Proposition 1: the figure-eight, run as a control STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
2. Theorem 2: butterfly I (I.A.1), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
3. Theorem 3: butterfly II (I.A.2), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
4. Theorem 4: bumblebee (I.A.3), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
5. Theorem 5: moth I (I.B.1), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
6. Theorem 6: moth II (I.B.2), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
7. Theorem 7: butterfly III (I.B.3), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
8. Theorem 8: moth III (I.B.4), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
9. Theorem 9: goggles (I.B.5), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
10. Theorem 10: butterfly IV (I.B.6), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE

11. Theorem 11: dragonfly (I.B.7), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
12. Theorem 12: yarn (II.B.1), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
13. Theorem 13: yin-yang I (II.C.2a), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
14. Theorem 14: yin-yang I (II.C.2b), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
15. Theorem 15: yin-yang II (II.C.3a), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
16. Theorem 16: yin-yang II (II.C.3b), a periodic orbit with its Morse index and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
17. Theorem 17: the orbit Hristov list #5, existence and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
18. Theorem 18: the orbit Li-Liao I.B_7 (Euler #2023), existence and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
19. Theorem 19: the orbit Li-Liao I.A_3 (Euler #4), existence and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
20. Theorem 20: the orbit Hristov list #9, existence and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE

21. Theorem 21: the orbit Li-Liao II.C_2 (Euler #64), existence and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
22. Theorem 22: the orbit Hristov list #10, existence and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
23. Theorem 23: the orbit Hristov list #29, existence STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
24. Theorem 24: the orbit Li-Liao II.C_61 (Euler #96), existence and linear stability STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
25. Theorem 25: the orbit Hristov list #67, existence STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
26. Theorem 26: the orbit Hristov list #87, existence STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
27. Theorem 27: the orbit that repeats the figure-eight word 17 times, period 116.22, existence STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
28. Theorem 28: the orbit that repeats the figure-eight word 18 times, period 85.49, existence STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
29. Theorem 29: the orbit that repeats the figure-eight word 11 times, period 55.91, existence STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
30. Theorem 30: the orbit that repeats the figure-eight word 10 times, period 77.42, existence STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE
31. Theorem 31: the orbit that repeats the figure-eight word 9 times, period 74.73, existence STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE

32. Theorem 32: the orbit that repeats the figure-eight word 10 times, period 72.92, existence STATUS: COMPUTER-ASSISTED PROOF; THE PROGRAM IS NOT PUBLISHED HERE

33. Proposition 33: the thirty orbits are pairwise distinct STATUS: CERTIFIED

34. Proposition 34: yin-yang II is linearly stable at both scales STATUS: CERTIFIED

35. Proposition 35: a second flow code on the fifteen rows, partial confirmation STATUS: CERTIFIED; THE PROGRAM IS NOT PUBLISHED HERE

36. Theorem 36: invariant two-tori accumulate at moth I STATUS: CONDITIONAL; THE PROGRAM IS NOT PUBLISHED HERE

37. Theorem 37: a KAM criterion applied to the seven hypotheses gives invariant tori STATUS: LEAN-CHECKED

38. Theorem 38: the twist determinant alone does not supply the other six hypotheses STATUS: LEAN-CHECKED

39. Lemma 39: the endpoint denominators of the twist determinant are positive STATUS: LEAN-CHECKED

40. Lemma 40: the endpoints of the twist determinant are in order STATUS: LEAN-CHECKED

41. Lemma 41: the upper endpoint of the twist determinant is negative STATUS: LEAN-CHECKED

42. Lemma 42: an outward lower bound on the twist determinant STATUS: LEAN-CHECKED

43. Lemma 43: an outward upper bound on the twist determinant STATUS: LEAN-CHECKED

44. Theorem 44: a periodic orbit at masses 1, 1, 3/2 that the catalogues checked do not list STATUS: CERTIFIED

45. Theorem 45: a linearly stable periodic orbit at masses 1, 1, 31/16 STATUS: CERTIFIED

46. Theorem 46: a linearly stable periodic orbit at masses 1, 1, 97/80, in the family of moth I STATUS: CERTIFIED
47. Open question: Morse index and KAM for the sixteen newer orbits STATUS: OPEN
48. Open question: Linear stability of three orbits from the published lists STATUS: OPEN
49. Open question: Formal identity of the six figure-eight-power orbits with catalogue entries STATUS: OPEN
50. Open question: Size of the tori around moth I, and slow drift STATUS: OPEN
51. Open question: A search of the rigorous-numerics literature for the sixteen STATUS: OPEN

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