Navier–Stokes: the original 2000 question: what is proved, and what is open

READING LEVEL KIDS · EVERYONE · EXPERTS

Each result is marked proved in writing, Lean-checked (a program that checks every step of a proof), cited, computed or open; the key results were also accepted by two separately written kernel checkers; not yet refereed.

proved in writing 35 · Lean-checked 23 · open 7 · cited 5 · computed; the program is not published here 1

CHECK IT YOURSELF

The check guide for this book · How to check a result yourself

---

What you can rerun: The Lean source (903 files, 898 of them Lean files) and the written proofs are from-files; a full Lean build takes hours and about 15 GB of memory. The computed sign pattern of the erratum is logs-only. Each result page gives its own rerun value and 5-minute check.

What a match shows: a match tells you that the published files really produce the published numbers. It doesn't tell you that the question was the right one to ask, or that a written argument around the numbers is complete. Each recipe says plainly what its steps show and what they don't. For a result whose evidence is logs (logs-only), a match checks the bookkeeping of the logs, not the search that wrote them.

Cite: Sonny (2026). Navier–Stokes: the original 2000 question: what is proved, and what is open (preliminary). Go check yourself. https://gocheckyourself.org/read/navier-stokes/everyone/

1. Start here

2. The simple version

3. The original question (2000): Navier-Stokes is not one problem
4. The updated question: the erratum
5. OpenAI, 2026: what it answered
6. The original 2000 wording: what Tao saw in 2011, and what follows
7. The other answers under the erratum
8. What remains open
9. A fidelity question raised by others
10. Every reading
11. Row splits
12. Exact statements
13. Theorems in this book
14. Check it yourself: Navier-Stokes, the four alternatives and the erratum
15. Files and fingerprints
16. Trust and verification
17. Lean sources: how to check them
18. People and works
19. Files and rights
20. Articles

1 Start here

Navier–Stokes: Four Questions, Every Reading, the Answers So Far · Sonny

What this book shows

- My result. Under the original 2000 wording, a fluid that starts at rest in the repeating box never has to break down, whatever allowed push acts on it: there is always a smooth flow that lasts forever, with a pressure that need not repeat. Proved in writing, with every step but one also checked in Lean (a program that checks every step of a proof), and accepted by two separately written kernel checkers, nanoda and lean4lean (different judges, written by different people, each given the proof as plain text); that the series in its H^3 bound (a measure of how smooth the flow stays) converges (its infinite sum settles to a finite value) is checked in writing.

- Why it matters. The 2000 wording requires the flow's velocity to repeat across the box, but sets no such condition on the pressure. So the problem as printed is already met by a flow from rest that never breaks down, and a breakdown that starts from rest cannot answer (D) as

printed (D is one of Clay's four questions, explained further on). For other starting states, (D) stays open.
- Clay's 2000 statement and its later correction, written out in Lean, with every reading of the clauses checked against them.
- Tao's link between (B) and (D) (Tao, 2011, cited), and (B) as the same question under both wordings, proved in Lean.
- Every theorem in this book is listed in the theorem library.

This begins with Terence Tao. In 2011 he noticed that the printed problem never asks the pressure to repeat, sketched what follows, and asked the question that became his Conjecture 1.8. I took his idea and carried it through: the case that starts from rest, for all time, with no extra assumption, proved in writing, with every step but one also checked in Lean (that the series in its H^3 bound converges is checked in writing), and an answer to his question that rests on OpenAI's construction.

Clay's problem, published in 2000, asks for one of four things, (A) to (D). Each asks whether a fluid that starts smooth stays smooth, with or without a push (an outside force), either in all of space or in a repeating box (a box whose opposite sides join up, so the flow repeats like wallpaper). This book takes the original 2000 wording at its word and answers each part as far as it can now be answered.

Under the original 2000 wording:

- (D) Repeating box, with a push: open as printed; from rest, it need not break down. From rest, every allowed push, OpenAI's included, has a smooth flow that lasts forever: the 2000 wording does not require the pressure to repeat, so the whole fluid can drift through the box. This second smooth solution does not contradict OpenAI, whose result is about flows whose pressure repeats. For other starting states, (D) stays open: on every finite time interval, (D) holds exactly when (B) fails, by Tao's Proposition 1.7 (2011); for all time this holds only if the version his Remark 6.2 sketches, with the details omitted, holds; cited, not rechecked in Lean. My idea: Tao's 2011 argument needed an unproven assumption, and for a fluid starting from rest that assumption holds

trivially. I proved the full statement, for all time and every allowed push, and checked every step but one in Lean; that the series in its H^3 bound converges is checked in writing.

- (C) All of space, with a push: it can break down. OpenAI (2026) gives a breakdown (not yet refereed). (C) and (D) are different settings: (C) is all of space, (D) the repeating box.
- (B) Repeating box, no push: open. It is the same question under Clay's later correction (observed by Tao, 2011; proved in Lean by Sonny).
- (A) All of space, no push: open.

Clay's correction. Clay later added an erratum (a published correction), undated, at the end of the problem statement; it also states that the pressure must repeat, and reprints Eqn (10). Under it, a breakdown from rest answers (D), and OpenAI's construction gives one: see the statement and the proof. Clay has not announced a decision on any of these answers, and no prize has been awarded (its Navier-Stokes page lists the problem as "Active").

The detail

Clay's Navier-Stokes prize problem asks about water and air in four settings. In (A) and (B) nothing pushes the fluid: does every smooth start stay smooth forever, in all of space (A) or in a box that repeats like wallpaper (B)? In (C) and (D) a push is allowed: can some smooth flow break down, in all of space (C) or in the repeating box (D)?

> **What I proved**
>
> I proved these results in Lean, against Clay's own clauses. Where noted, nanoda and lean4lean, two separately written programs, checked Lean's work again.
>
> - Under Clay's 2000 wording, from rest, every push has a smooth flow that lasts forever: the answer to (D) depends on the wording. The fluid slides through the box, with a pressure that tilts across space. Printed (D) asks the solution only for (1), (2), (3), (10) and (11): the equations, the start, a repeating velocity and smoothness, with no

energy bound. The slide meets these, and in the Lean statement its speed and the energy in each cell also stay bounded for all time. There is no contradiction with OpenAI's result: same push and same start, but a different solution, whose pressure does not repeat. My idea: starting from rest, Terence Tao's 2011 slide (Proposition 1.7; for all time, sketched in Remark 6.2 with the details omitted) needs no assumption, because the resting fluid is already an unforced solution, so it applies to every push in Clay's class, OpenAI's included. I wrote a complete proof, for all time, of the case from rest, and checked every step but one in Lean (nanoda, lean4lean); its H^3 bound is stated in Lean, but without the clause that its series converges; that step is checked in writing. Chapter 4

- (A) and (B) under readings that add a condition the text does not contain, such as finite energy over all of space, a smooth extension before time zero, zero average velocity or an extra integral identity. I decided most of these readings, in Lean and nanoda for most of them and in writing for the rest. Those still open are stated exactly. Chapter 8 gives all 440,401,920 readings, each with its verdict and basis.
- The construction, tested against every reading. I tested OpenAI's push from rest against every reading of (D) in chapter 8. It answers (D) where flow and pressure must both repeat, or where an added condition forces that or settles (D) by itself. It does not answer (D) under the 2000 wording from rest, nor under most readings with weak solutions (solutions that are allowed to be rough), a pressure repeating only at time zero, a smooth zero extension or one push for every viscosity; chapter 8, section 4 lists each, the open ones included. Lean for the main cases; written or cited for the rest.
- OpenAI's Lean package, checked against Clay's clauses. I ported the package to a pinned toolchain (a fixed set of software versions), rebuilt it with only the standard axioms (starting assumptions Lean accepts without proof), and checked its statements of (C) and (D) against Clay's wording. nanoda accepted both; lean4lean accepted (D). Chapter 3

- (D) under the correction read word for word (the flow itself need not repeat). Without repeating pressure, the same slide works, so no breakdown from rest answers (D). With repeating pressure, (D) is open for OpenAI's push. I proved that, for any repeating push, a competing flow growing at most linearly in space must repeat (Sonny (2026); Lean, nanoda). I then proved that any escape for OpenAI's push must grow faster than linearly (far away, its speed outruns every fixed multiple of the distance) before the breakdown time (Sonny (2026); Lean, nanoda). Chapter 6

Checked in Lean against Clay's wording: (B) is the same question under both wordings (Tao, 2011, p. 6 and Lemma 4.1); proved in Lean against Clay's clauses by Sonny. With no push, a smooth flow with repeating velocity exists from a given start exactly when one with repeating velocity and repeating pressure exists. Chapter 5

Tao's 2011 question, Conjecture 1.8, now has an answer:
no. I answered it, and checked it in Lean for every period.

A conjecture is a careful guess, posed so that others can test it. Tao proposed this one in 2011 as a repaired form of Clay's box problem. The answer is no. I showed, and proved in Lean for every period, that OpenAI's Corollary 10.6 answers Tao's Conjecture 1.8 (the answer is no): by Tao's footnote 4, normalised pressure is periodic pressure up to a constant, and Corollary 10.6 rules out a smooth solution with periodic pressure. Two short written steps connect the Lean definitions to Tao's words (the Conjecture 1.8 written steps). Start from rest with OpenAI's push at viscosity 1, which is smooth and repeats with period 1. For every end time T of 1 or more, there is no smooth periodic solution on [0, T] with normalised pressure (a pressure that repeats with the flow). This holds under each reading of Tao's normalisation and whatever period the solution has. Corollary 10.6 is OpenAI's own Lean theorem, rebuilt and checked in this book. The unforced case, Tao's Conjecture 1.4, is open. Sonny (2026); Lean, nanoda. Chapter 4

> **What OpenAI proved**
>
> - (C), all of space, with a push. The OpenAI team built the construction (OpenAI, 2026), in which a smooth push drives a fluid to break down at time 1. The mathematics of the construction is OpenAI's.
> - (D) under the correction as intended, from rest: can break down, yes, by the same construction, with flow and pressure both repeating. OpenAI's README describes their results as alternatives (C) and (D) "in the Clay Mathematics Institute's official problem description". Chapter 3

> **What this means for the Clay prize**
>
> The Clay Institute decides. Its wording leaves room for interpretation, which is why this book gives the answer under every reading, and the Institute may weigh every reading, the results here included. The classical questions, (A) and (B) with nothing added, are still open. (C) is answered yes (OpenAI, above; not yet refereed).

The question

Water and air move by the Navier-Stokes equations. Can a smooth flow, starting calmly, break down in finite time, or does it always carry on smoothly? In 2000 the Clay Mathematics Institute named this one of its seven Millennium Prize Problems, and Charles Fefferman wrote the official description. A proof of any one of its four alternatives, (A) to (D), answers the problem. The description says, for alternative (A):

> "(A) Existence and smoothness of Navier-Stokes solutions on R^3. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x, t)$ to be identically zero. Then there exist smooth functions $p(x, t)$, $u_i(x, t)$ on $R^3 \times [0, \infty)$ that satisfy (1), (2), (3), (6), (7)."

(C. L. Fefferman, Existence and smoothness of the Navier-Stokes equation, as hosted by the Clay Mathematics Institute. The other three alternatives are

quoted in the exact statements page.)

Two doors say "it always works": start the flow calmly, give it no push, and it stays smooth forever. Two doors say "it can break": some calm start plus some push makes the flow break down. Each kind comes in two places, all of space or a box that repeats like wallpaper.

	No push: it always works	**With a push: it can break**
All of space	(A) every calm start stays smooth forever	(C) some start and push make it break down
A box that repeats like wallpaper	(B) every calm start stays smooth forever	(D) some start and push make it break down

A few sentences in the text can be read more than one way: whether the pressure repeats too, what "smooth" and "finite energy" mean, and whether weak or strong solutions are meant. Chapter 1 lays the four alternatives out.

The description now ends with an erratum, which says: "The further condition p(x + e_j, t) = p(x, t) should be made explicit in Eqn (8)." That adds a requirement that the pressure repeat in the box. It also says "Eqn (10) should read:" and prints a corrected equation under a label that does not match it. Chapter 2 gives the exact text and the ways it can be read. The question this book answers is: under every reading the wording allows, of the 2000 text and of the erratum, which of the four alternatives is answered yes, which is answered no, and which is still open, and how was each answer checked?

Why it matters

The Navier-Stokes equations are the standard model of flowing water and air. Whether their smooth solutions can break down is one of the most famous open questions in mathematics, and the Clay Institute offers a prize for settling it. Whether a claimed answer counts depends on the exact wording of the problem, and the wording can be read in more than one way. A claim of an answer is only as good as the reading it is made under.

What answering it would help with

- A table of verdicts under every reading lets later papers say plainly which wording and which alternative they address.
- Knowing which readings leave the question open points work at the cases that are still undecided, such as (A) and (B) in their classical form.
- The Lean statements, kept beside the plain-words descriptions, let readers judge how closely a formal result matches the English problem (see chapter 7).

What our research shows

How to read the answers. Each row says what its yes means. For (A) and (B): stays smooth, yes or no. For (C) and (D): can break down, yes or no. Open means nobody knows yet. "From rest" means the answer is given for a start from rest. Where (A) or (B) is no under a reading that adds a condition, no smooth solution exists under that reading; this comes from the added condition, not from a breakdown. A "no" is a proved result, just as a "yes" is.

What this is: The answer key: for each way of reading Clay's 2000 Navier-Stokes problem text, and for the erratum wording, the verdict on each of the four alternatives (A) to (D), who answered it, the reason, and how it was checked (Lean, a separately written proof checker, a written proof, or a classic theorem we cite).

The answer key

[figure: The four alternatives under the 2000 wording and under the correction as intended]

Each answer carries its basis: Lean, written (a complete written proof in the book), or cited (a classic theorem, not rechecked here). In each cell: the answer, then who answered it, then what checked it. The same rows in full, with the sentence each reading changes, are in the plain words page and chapter 8.

Question	The 2000 wording (the pressure may tilt)	The correction, as intended (the pressure must repeat too)
(A) all of space, no push: does every smooth start stay smooth?	Open: the classical question	Open: the classical question
(B) repeating box, no push: does every smooth start stay smooth?	Open: the classical question	Open: the classical question
(C) all of space, with a push: can some smooth flow break down?	Can break down: yes (OpenAI, 2026; Lean and nanoda, rebuilt and checked by Sonny)	Can break down: yes: OpenAI's construction (OpenAI, 2026), checked under this wording by Sonny (Lean)
(D) repeating box, with a push: can some smooth flow break down?	From rest: every push has a smooth flow that slides through the box forever (Sonny, 2026; proved in writing, with every step but one also checked in Lean, nanoda and lean4lean; that the series in its H^3 bound converges is checked in writing). Other starts: open; on every finite time interval, (D) holds exactly when (B) fails, by Tao's Proposition 1.7 (2011); for all time this holds only if the version his Remark 6.2 sketches, with the details omitted, holds; cited, not rechecked in Lean	Can break down: yes, from rest (OpenAI, 2026; Lean, nanoda, lean4lean, rebuilt and checked by Sonny)
Tao's Conjecture 1.8 (2011), with a push, in Tao's own wording (normalised pressure, which repeats)	Answered: no (Sonny, 2026; Lean, nanoda)	Answered: no: the same result, which does not depend on Clay's wording
Other readings, which add a condition the text does not contain	Energy read over all of space: (D) can break down: yes; (B) stays smooth: no (Sonny, 2026; Lean, nanoda). An integral identity as an extra rule: (D) can break down: yes; (B) and (A) stay smooth: no (Sonny, 2026; Lean, nanoda, for the text's test fields). Weak solutions only, no extra identity: (A) and (B) stay smooth: yes; (C) and (D) can break down: no (Leray 1934, Hopf 1951; cited). Decay imposed inside the box: only the zero flow is left, so (B) stays smooth: yes and (D) can break down: no (Sonny, 2026; written). Where (A)	The same answers for these four readings

Question	The 2000 wording (the pressure may tilt)	The correction, as intended (the pressure must repeat too)
	or (B) stays smooth: no here, no smooth solution exists under that reading; this comes from the added condition, not from a breakdown. All 440,401,920 readings, each with its verdict and basis, are in chapter 8 (Sonny, 2026)	

Other starts under the 2000 wording (Terence Tao, 2011): on every finite time interval, (D) holds exactly when (B) fails, by Tao's Proposition 1.7 (2011); for all time this holds only if the version his Remark 6.2 sketches, with the details omitted, holds; cited, not rechecked in Lean. Tao's Proposition 1.7 gives this on every finite time interval, and his Remark 6.2 sketches the version for all time, with the details omitted (cited, not rechecked in Lean).

The correction, read word for word (its label "Eqn (10) should read" replaces the rule that the flow repeats): (A) is open; (B) is open, reduced to a system for the velocity alone (Sonny, 2026; written); (C) unchanged; (D) from rest without repeating pressure: every push has a smooth flow, by the same slide (Sonny, 2026; Lean, nanoda, lean4lean); (D) with repeating pressure: open for OpenAI's push, and any escape must grow faster than linearly in space (Sonny, 2026; Lean, nanoda).

Some readings add a condition the text does not contain. A few examples, each from chapter 8:

- Energy read over all of space, even inside the box. (D) can break down: yes; (B) stays smooth: no (Sonny, 2026; Lean and nanoda, stated directly against Clay's 2000 wording).
- An integral identity treated as an extra rule. (D) can break down: yes; (B) stays smooth: no, because explicit starting flows have no solution (Sonny, 2026; Lean and nanoda for the text's test fields; written for other test conventions). Under the 2000 wording, with the identity imposed on (A) itself, the text's test fields and solutions smooth up to time zero, (A) stays smooth: no too (Lean and nanoda).

- Weak solutions only. (A) and (B) stay smooth: yes; (C) and (D) can break down: no, from the theorems of Leray (1934) and Hopf (1951) (cited), unless an extra identity is also imposed.
- Decay in space imposed inside the box, a condition the text gives only for all of space. Only the zero flow is left: (B) stays smooth: yes; (D) can break down: no.

Chapter 8 gives every reading its verdict and counts how many of the 440,401,920 readings now have a proved answer; each reading counts once there, so the shares say nothing about which readings are reasonable.

The article

A shorter, personal telling of all of this is in the article The Pressure Loophole: How the Wording Decides the Answer.

Experts: the files and checks are here: the check guide (every count, checker and export size), the exact statements page and the certificate list.

Yesterday, today, tomorrow

Yesterday. In 2000 Charles Fefferman wrote the official Navier-Stokes problem for the Clay Mathematics Institute, with four alternatives, (A) to (D). Clay's description now ends with an erratum that adds periodic pressure; when it was added is not established. In 2011 Terence Tao noticed that for the periodic box the printed wording asks the velocity to repeat but does not say the same of the pressure, and he proposed a repaired form. In 2026 the OpenAI team published a forced breakdown in the periodic box, with periodic pressure, together with a Lean package, building on work of Córdoba and Martínez-Zoroa.

Today. This book reads the problem text in every way its wording allows and gives each reading's verdict on (A) to (D), with who answered it and the basis. My results are in the box "What I proved", above. The sources we cite each state the wording they use; a table of verdicts under every reading, set against Clay's printed clauses, was not among them, and it is what this book adds.

Tomorrow. Reproduction needs the pinned Lean toolchain, the complete theorem and dependency sources, and the commands in the check guide. The four-question guide in chapter 8 lets later papers say which wording they use.

Check it yourself

- The check guide: what is claimed, the exact commands to rebuild and recheck it, the expected output for each, a recipe for recounting the reading combinations, the limits of the checks, and what is not claimed.
- The exact statements page: Clay's clauses and the Lean statements, word for word.
- The book's Lean folder holds the Lean sources, with a pinned toolchain and lockfile (a fixed list of exact software versions). The certificate list lists the declarations behind each cell of the answer key.

The Lean names are internal labels, not terms of the problem. The book describes each result in plain words instead. Those names appear only in code blocks, so that you can run the checks.

How I direct this work, and my own tools, are not part of this book. Everything you need to check the results is public: the results, their certificates and fingerprints, the Lean proofs, and a written recipe for each check.

How this research was done: https://gocheckyourself.org/about/#how

Start here, by reader

- The simple version, for everyone: the plain words page. Ten minutes, no equations.
- For younger readers: the children's version.
- The scientific version, for specialists: the answer key, then chapters 1 to 8 below, then the check guide to check it yourself.

The book

- Chapter 1: The original question (2000): Navier-Stokes is not one problem
- Chapter 2: The updated question: the erratum
- Chapter 3: OpenAI, 2026: what it answered
- Chapter 4: The original 2000 wording: what Tao saw in 2011, and what follows
- Chapter 5: The other answers under the erratum
- Chapter 6: What remains open
- Chapter 7: A fidelity question raised by others
- Chapter 8: Every reading, with the row splits and their proofs
- The article

Credit

We stand on the shoulders of giants: Charles Fefferman and the Clay Mathematics Institute, Terence Tao, the OpenAI team, Córdoba and Martínez-Zoroa, an anonymous reader of Tao's blog, the Formal Conjectures authors, Bastounis, Circelli and Hansen, Koch, Nadirashvili, Seregin and Šverák, Fushchych and Popovych, A. N. Tychonoff, J. Leray, E. Hopf, R. T. Seeley, Fujita and Kato, Giga and Miyakawa, Lemarié-Rieusset, Cannone, Temam, and Bruè and De Lellis. See the credits page. Licence: the licence note; inherited files are Apache-2.0 (the notice file).

Go check yourself.

Where to go next

- Why does one sentence about pressure change the answer for (D)? See The original 2000 wording: what Tao saw in 2011, and what follows.
- What exactly is still undecided, and what would decide it? See What remains open.
- Which of the more than 440 million readings applies to a given claim? See Every reading.

Have a question about this? Please get in touch.

2 The simple version

The simple version

Ten minutes, no equations. Each part has a short in plain words section and, where it helps, a short for experts note you can skip.

This begins with Terence Tao. In 2011 he noticed that the printed problem never asks the pressure to repeat, sketched what follows, and asked the question that became his Conjecture 1.8. I took his idea and carried it through: the case that starts from rest, for all time, with no extra assumption, proved in writing, with every step but one also checked in Lean (a program that checks every step of a proof); that the series in its H^3 bound converges is checked in writing, and an answer to his question that rests on OpenAI's construction.

The question

In plain words. The Navier-Stokes equations say how water and air move: how fast each bit of fluid goes, and how the pressure pushes it around. They work very well in practice. What nobody has shown is whether they always behave. Could a smooth, calm flow ever develop a point where its speed becomes infinite? That moment is called a breakdown.

In 2000 the Clay Mathematics Institute made this one of its seven Millennium Prize Problems. Charles Fefferman wrote the official question.

For experts. Incompressible Navier-Stokes in three dimensions, viscosity $\nu > 0$, on all of space or on the periodic box; smooth solutions for all time versus finite-time blowup. Fefferman's official description, equations (1) to (12).

Four doors

In plain words. Clay's 2000 text offers four doors, (A) to (D). Settle any one and the problem is answered.

- Two doors say "it always works": give a calm flow no push, and it stays smooth forever.

- Two doors say "it can break": you may choose a push, and some calm start plus that push makes the flow break down.
- Each kind comes in two places: all of space, or a box that repeats like wallpaper (space cut into identical tiles, with the flow the same on every tile).

	No push: it always works	**With a push: it can break**
All of space	(A) every calm start stays smooth forever	(C) some start and push make it break down
A box that repeats like wallpaper	(B) every calm start stays smooth forever	(D) some start and push make it break down

In the "can break" doors the solver chooses the push. In the "always works" doors nobody does: it has to hold for every calm start.

For experts. (A) and (B) are global regularity without forcing; (C) and (D) are blowup with a forcing term chosen by the solver. See chapter 1.

Sentences that can be read more than one way

In plain words. The 2000 text is careful, but a few sentences can be read in more than one way. Each reading changes what the four doors mean.

- Does the pressure repeat? In the box, the flow must be the same on every tile. The printed text does not say the pressure must be. If the pressure may rise steadily from one side of the floor to the other, like a tilted floor, the whole fluid can slide across the tiles. The tilt takes up the push's average, and sliding fast makes the rest of the push average out, so the fluid need not break down. Terence Tao pointed this out in 2011.
- What counts as "smooth"? Smooth up to the starting moment, or only after it? Smooth for ever, or a "weak" solution that is allowed to be rough?
- What counts as "finite energy"? The text asks for bounded energy over all of space in (A) and (C), and says nothing of the kind for the box. Some readers carry the condition over.

- Weak or strong solutions? Weak solutions always exist (Leray 1934, Hopf 1951), so a reading that allows them changes the answers sharply.
- What do the erratum's two sentences say? Clay's page now ends with an erratum. It asks that the pressure repeat, and it prints a corrected integral equation under the label "Eqn (10)", although (10) is the rule that the flow repeats. Read as intended, the flow repeats and the pressure repeats. Read word for word, the flow no longer has to repeat.

Chapter 8 lists sixteen such clauses and 440,401,920 ways of combining the readings. Most do not change the answer once a few others are fixed, so they fall into a short list of cases.

For experts. The erratum asks for the periodicity "made explicit in Eqn (8)" and labels a corrected weak identity "Eqn (10)". Chapter 2 gives the exact text; chapter 4 covers Tao (arXiv:1108.1165, 2011).

OpenAI's construction

In plain words. In 2026 the OpenAI team published a force that drives a fluid, starting perfectly still, to a breakdown at time 1, in the repeating box, with the pressure repeating too. They wrote that they aimed at (D) "as required in the erratum". They also formalised the periodic result in Lean. They credit Córdoba and Martínez-Zoroa for the strategy behind it.

For experts. Finite time blowup for Navier-Stokes (OpenAI, 2026), with the Lean package NavierStokesAndEuler (Apache-2.0). Chapter 3.

Based on the original 2000 wording, here are the answers

In plain words. Each line gives the reading, what it adds to the text (if anything), the verdicts, and a tag for the basis:

- Lean: checked by Lean, with the separate checkers named.
- written: a complete proof in the book (the limits of review are in the check guide).
- cited: a classic theorem, not rechecked here.

Each verdict is one of three. Answered: yes means the alternative is proved true under that reading. Answered: no means it is proved false under that reading. Open means nobody knows yet. A "no" is a proved result, just as a "yes" is.

The 2000 text with nothing added.

- (A) all of space, no push: open. This is the classical question.
- (B) repeating box, no push: open. The classical question again.
- (C) all of space, push: answered: yes. OpenAI's whole-space construction meets the ordinary printed (C), checked in Lean and nanoda (a second checker, written by different people, that is given the proof as plain text; the whole-space breakdown).
- (D) repeating box, push: from rest, every push has a smooth flow (Sonny, 2026; proved in writing, with every step but one also checked in Lean; that the series in its H^3 bound converges is checked in writing). A fluid at rest can always slide away, with a tilted pressure, whatever the push (the slide-through escape: Lean, nanoda, lean4lean, another such second checker), so no breakdown from rest meets it. Other starts: open; on every finite time interval, (D) holds exactly when (B) fails, by Tao's Proposition 1.7 (2011); for all time this holds only if the version his Remark 6.2 sketches, with the details omitted, holds; cited, not rechecked in Lean.

- With no push at all and a fluid at rest, the printed class already allows two different solutions, rest and a steady slide (Sonny, 2026, following a remark of Tao; the conditional zero-force drift and the bounded drift: Lean).

If the pressure must repeat at the starting moment only (this adds a condition on the starting data). (D) stays open. Tilt removal while the force is zero (Sonny, 2026; written, using a short time in which OpenAI's force is zero, which is checked in Lean) gives a competitor, so OpenAI's construction does not meet it.

If the text's energy condition is read over all of space, even in the box (this adds a condition the box alternatives do not list). (D) holds (Sonny, 2026; Lean and nanoda, stated directly against the 2000 wording; the whole-space energy exclusion is a second, written route), because a push that swirls has no solution with finite energy over all of space. OpenAI's construction meets it. (B) fails (Sonny, 2026): a periodic start that is not zero has infinite energy over all of space.

If an integral identity is treated as an extra rule (a condition on averages of the flow: the printed identity (12), or a half-corrected one; no alternative lists it). (D) holds and (B) fails (Sonny, 2026; the extra-identity reduction, the frozen-vorticity obstruction and the shear-start obstruction; Lean and nanoda for the text's test fields, written for other test conventions): explicit starting flows have no solution at all. These readings settle (D) for reasons unrelated to blowup, so they lend OpenAI's construction no support.

If decay in space, as in (4) and (5), is imposed inside the box (the text gives it only for all of space). Only the zero start and zero push are left, so (D) fails and (B) holds (Sonny, 2026). OpenAI's force is outside this class.

If only weak solutions are asked for (the text asks for "smooth functions"). (A) and (B) hold; (C) and (D) fail (Sonny, 2026, from the cited theorems of Leray 1934 and Hopf 1951). This changes if an extra identity is also imposed; see the last section.

If the average velocity must be zero at every time. (D) holds (Sonny, 2026; Lean and nanoda, stated directly against the 2000 wording; the witness forces for (D)). (B) fails for starts with a nonzero average (Sonny, 2026; Lean and nanoda).

If everything is smooth across the starting moment, with a zero extension before it. (D) holds and (A) and (B) fail (Sonny, 2026; Lean and nanoda, stated directly against the 2000 wording). (C) holds (Sonny, 2026; the witness forces for (D), on all of space; written).

If smoothness is asked only after the starting moment. The same answers as without it, apart from one corner ((A) with the pressure repeating at the starting moment only and smoothness only afterwards, below).

If the pressure may grow only slower than linearly. For a repeating flow this is the same as "the pressure repeats" (Sonny, 2026; the pressure-periodicity lemma; Lean).

If an energy inequality on a cell (one tile of the box) is added. Sonny (2026) settled part of this reading: when the flow repeats and the push is chosen for each viscosity (how sticky the fluid is) or for one viscosity, even with a tilted pressure, OpenAI's construction meets it; when one push must serve every viscosity, it does not. The remaining cases (a flow that need not repeat, or one push for every viscosity with no pressure gradient) are open (chapter 8, section 4).

Based on the erratum wording, here are the answers

The erratum as intended (the flow repeats, the pressure repeats, the corrected identity adds nothing; see the smooth-solutions identity, Sonny's (2026), in Lean and nanoda for solutions smooth up to the starting moment, for why).

- (D) holds. OpenAI's construction meets it, from rest. For every viscosity there is a push for which no smooth, from-rest flow with repeating velocity and pressure lasts for ever (the forced breakdown with repeating pressure: Lean, nanoda, lean4lean). The same conclusion over broader classes of pushes: its three wider-class statements (Lean, nanoda, lean4lean). The tilt is what the printed wording leaves open: once the pressure must repeat, the universal slide is false (Sonny, 2026; the escape-pressure criterion and the failure of the escape under repeating pressure: Lean).
- (A) and (B): open (the classical questions). (C): answered: yes within the erratum's stated equations, as above (OpenAI's whole-space breakdown, 2026).

The erratum word for word, with the pressure required to repeat (the flow need not).

- (D): open, narrowed by Sonny (2026). Without the flow having to repeat, a wild flow, different on every tile and growing very fast far away, might escape OpenAI's push. Any flow that grows at most linearly must repeat, so the forced breakdown with repeating pressure excludes it; so any escape for OpenAI's push must grow faster than linearly with distance (both Lean and nanoda, chapter 6). Every escape carries a nonzero "period defect" that solves a simpler system (Lean, nanoda; the Lean sources are in the book). Eleven explicit attempts to build one fail for stated reasons (chapter 6).

The erratum word for word, with no pressure requirement.

- (D) from rest: every push has a smooth flow (Sonny, 2026): the printed class lies inside this one, so the slide of the slide-through escape applies (Lean, nanoda, lean4lean).

The erratum applied to all of space as well (the erratum names only (8) and (10), which (A) and (C) do not cite). If the pressure must repeat in all of space, (A) fails and (C) holds (Sonny, 2026; the pressure-on-all-of-space lemma: Lean and nanoda for solutions smooth up to the starting moment, and for solutions smooth after it with the pressure repeating at every later moment; written for the rest. The idea: a periodic quantity that fades out far away must be zero).

The few readings where nothing is answered

Three groups of readings stay open as the classical questions. They are listed here so no one has to hunt for them. The other open groups, mostly readings that add a condition, are listed in chapter 8, section 4.

- (A), the classical (A) question and the other open readings in its row of chapter 8: 135,659,520 combinations. With the 2,359,296 open readings of (A) with weak solutions and the erratum applied to all of space, this

group is the open part of (A). Besides the text with nothing added, this count holds readings that add a condition and are still counted open.

- (B), flow repeats, nothing added (the classical (B) question): 39,739,392 combinations. The classical question.
- (D) statement, all other readings (the remaining readings of (D)): 26,062,848 combinations. No witness known; for the printed wording this is the question whether (B) fails (Tao 2011, Proposition 1.7).

Counting the readings

I took every reading of Clay's text, 440,401,920 combinations, and gave each one its answer. The share of readings answered and still open for each door is in chapter 8, with its caveat: the shares count readings equally and say nothing about which readings are reasonable. The classical questions, (A) and (B) read with nothing added, are still open. Every reading and its answer is in chapter 8.

Every reading has a label (answered: yes, answered: no or open), and large groups remain open; chapter 8 lists each one with its count.

Groups of readings split into parts

Some groups of readings split into parts that are answered: yes, answered: no and open. Each split, with its count and its basis, is in chapter 8, section 4, and the reading splits.

How many theorems, and how they are checked

In plain words. The results rest on 12 statements checked in Lean, collections of Lean helper statements, and 15 lemmas in chapter 8. Lean uses the three standard axioms that ordinary Lean mathematics relies on. The key results were checked again by two other proof checkers, written by different people: nanoda and lean4lean. Every row says whether it rests on Lean, a written proof or a cited theorem.

Why this title

In plain words. Technically correct is the best kind of correct. Here it is meant literally. The Navier-Stokes question was written with great care, and its answer can turn on one sentence about pressure. The task here was to read that sentence every way it can be read, and say exactly what is true under each. Nothing here takes anything away from the work it builds on.

We stand on the shoulders of giants. See the credits page.

Which wording Clay meant is for Clay to say. Read the rows, decide which wording you think governs, and check everything yourself: the check guide.

Where to go next

- Why does a sentence about pressure matter so much? See The original 2000 wording: what Tao saw in 2011, and what follows.
- What exactly is still open? See What remains open.

Yesterday, today, tomorrow

Yesterday: In 2000 Charles Fefferman wrote down the official question about how water and air behave, for the Clay Mathematics Institute, as four alternatives. Clay's page now also carries a correction about pressure; when it was added is not known. In 2011 Terence Tao noticed that one sentence about pressure could be read in more than one way. In 2026 a team at OpenAI built a push that makes a calm flow break down in the repeating box when the pressure must repeat too, and shared a Lean version of it. Lean is a program that checks every step of a proof.

Today: I proved that under the original 2000 wording a fluid that starts at rest in the repeating box need not break down, whatever the push: it can always slide away smoothly, forever (proved in writing, with every step but one also checked in Lean, nanoda and lean4lean; that the series in its H^3 bound, a measure of how smooth the flow stays, converges, is checked in writing). I also read the problem text in every way its wording allows. For each reading, the book says what follows for each of the four alternatives, and

how that was checked. Below you will find the answers under the original 2000 wording, then under the erratum, then the rows where nothing is answered.

Tomorrow: Other people can rerun the checks themselves. The guide in chapter 8 lets others say which reading of the problem they mean.

3 The original question (2000): Navier-Stokes is not one problem

The original question (2000): Navier-Stokes is not one problem

The question

What exactly did Clay's 2000 text ask, and in how many ways can its four alternatives be read?

Why it matters

A claimed answer is only as good as the reading of the question it is made under, and some sentences of the 2000 text can be read more than one way.

What answering it would help with

Naming the four alternatives and the sentences that vary lets later work say plainly which question it answers.

What our research shows

In plain words

The Navier-Stokes equations describe how a fluid such as water or air moves. In 2000 the Clay Mathematics Institute named their behaviour one of its seven Millennium Prize Problems. Charles Fefferman wrote the official problem description.

People often speak of "the" Navier-Stokes problem. The official description actually offers four alternatives, labelled (A) to (D). A proof of any one of

them answers the problem.

	Where the fluid lives	Is a force pushing it?	What you must show
(A)	all of space	no	Every smooth, incompressible starting flow that fades out far away carries on smoothly forever, with bounded total energy.
(B)	a periodic box (space that repeats like floor tiles)	no	Every smooth, incompressible periodic starting flow carries on smoothly forever.
(C)	all of space	yes, one that fades out	Some smooth start and some such force make the flow break down: no smooth solution with bounded total energy lasts forever.
(D)	a periodic box	yes, periodic and fading	Some smooth start and some such force make the flow break down: no smooth periodic solution lasts forever.

So there are two questions, each asked in two settings:

- Does a smooth flow always carry on? That is (A) on all of space and (B) in the periodic box.
- Can a smooth flow be forced to break down? That is (C) on all of space and (D) in the periodic box.

(A) and (B) are about flows with no force. (C) and (D) allow a force chosen by the solver. That difference matters. A result about (C) or (D) says nothing about (A) or (B): they are different questions, even though Clay's rules count a proof of any one alternative.

The rest of the book takes the question in this order: the erratum that updated it (chapter 2); what OpenAI answered (chapter 3); the original 2000 wording, what Tao saw in 2011 and what follows (chapter 4); the other answers under the erratum (chapter 5); what remains open (chapter 6); a fidelity question raised by others (chapter 7); and every reading in full (chapter 8). Chapters 2 to 7 are mostly about (D), the periodic breakdown alternative, and about one detail of its wording: what it says, or does not say, about the pressure. Chapter 8 covers all four alternatives under every reading.

Why the pressure matters

A fluid flow has two unknowns:

- its velocity, how fast and in which direction each bit of fluid moves;
- its pressure, which pushes the fluid around.

In the periodic box, the velocity must repeat from tile to tile. The question here is whether the pressure must repeat too. As we will see, the printed description requires it of the velocity only. A later erratum adds it for the pressure. Chapters 2 to 4 tell that story; the answer key shows what follows from it.

Yesterday, today, tomorrow

Yesterday. Charles Fefferman wrote the official description of the Navier-Stokes problem in 2000, and the Clay Mathematics Institute made it one of its seven Millennium Prize Problems. His text offers four alternatives, (A) to (D), and a proof of any one of them answers the problem.

Today. This chapter lays the four alternatives side by side: (A) and (B) ask whether a smooth flow always carries on with no force, on all of space and in the periodic box; (C) and (D) ask whether a chosen force can make a flow break down, in the same two settings. It points out that a result about (C) or (D) says nothing about (A) or (B): they are different questions, even though Clay's rules count a proof of any one alternative. Chapters 2 to 7 concern (D) and the wording about pressure; chapter 8 covers all four alternatives.

Tomorrow. A reader can use the table to see which alternative a given result speaks to. The same table lets future work say plainly which of the four it addresses.

For specialists

The official description (Fefferman, page 2) poses the equations (1)-(3) for velocity u and pressure p with viscosity $\nu > 0$, force f and initial velocity u°. The four alternatives refer to these numbered conditions:

	Data conditions	Solution conditions	Quantifier
(A)	u° smooth, divergence-free, satisfying (4); f ≡ 0	(1), (2), (3), (6) smoothness, (7) bounded energy	for all u° there exist p, u
(B)	u° smooth, divergence-free, satisfying (8); f ≡ 0	(1), (2), (3), (10), (11)	for all u° there exist p, u
(C)	u° and f satisfying (4), (5)	no (p, u) with (1), (2), (3), (6), (7)	there exist u°, f
(D)	u° and f satisfying (8), (9)	no (p, u) with (1), (2), (3), (10), (11)	there exist u°, f

The periodic conditions, as printed:

- (8): $u^\circ(x + e_j) = u^\circ(x)$ and $f(x + e_j, t) = f(x, t)$ for $1 \le j \le n$. These are conditions on the data.
- (9): $|\partial_x^\alpha \partial_t^m f(x, t)| \le C_{\alpha m K} (1 + |t|)^{-K}$ on $\mathbb{R}^3 \times [0, \infty)$, for any α, m, K.
- (10): $u(x, t) = u(x + e_j, t)$ on $\mathbb{R}^3 \times [0, \infty)$ for $1 \le j \le n$. Periodicity of the velocity.
- (11): $p, u \in C^\infty(\mathbb{R}^n \times [0, \infty))$. Smoothness of both fields; no periodicity of p.

The periodic alternatives (B) and (D) carry no energy clause like (7). As printed, neither (10) nor (11) asks the pressure to be periodic, so a solution's pressure may include a part that grows linearly in x (an affine part). The exact clause text is in the exact statements page.

The erratum names only (8) and (10), which (A) and (C) do not cite, so the whole-space alternatives read the same before and after it. Chapter 8 also covers the reading that applies the erratum to all of space.

Check it yourself

Clay's clauses and the Lean statements are quoted word for word in the exact statements page, and the check guide gives the commands to rebuild and recheck them.

Where to go next

- Why does a sentence about the pressure change the answer for the periodic box? See The original 2000 wording: what Tao saw in 2011, and what follows.
- What did the later erratum change? See The updated question: the erratum.

4 The updated question: the erratum

The updated question: the erratum

The question

What does the erratum at the end of Clay's description say, and in which ways can it be read?

Why it matters

The erratum adds a pressure requirement and prints a corrected equation under a label that does not match it, and the answer for (D) depends on how it is read.

What answering it would help with

Separating the reading as intended from the reading word for word lets others say which one a claim is made under.

What our research shows

In plain words

The official problem description, as Clay hosts it today, ends with a page headed Errata. It makes two changes.

1. Pressure must repeat. It adds the condition that the pressure repeat from tile to tile in the periodic box.

2. A corrected equation. It prints a corrected version of one integral identity.

Both changes are reasonable, but each leaves something to interpret.

- The added pressure condition is put in a list of starting conditions. The erratum says the condition "should be made explicit in Eqn (8)". Equation (8) lists conditions on the starting flow and the force, the things a solver chooses at the outset. Pressure is not chosen at the outset; it is part of the solution. So the erratum clearly wants periodic pressure, but it does not say in words that the solutions in (B) and (D) must have it. The natural reading is that they must.
- The corrected equation carries a different number. The erratum says "Eqn (10) should read:" and then prints an identity. Equation (10) is the condition that the velocity repeats. The printed identity is equation (12), the weak form of the equations, with two signs corrected. So the label "(10)" looks like a slip for "(12)".

Taken together, the sensible reading is: keep velocity periodicity (10), correct the signs in (12), and require periodic pressure of the solutions. This book calls it the erratum reading. The answer key uses it in its middle column.

Read word for word, "Eqn (10) should read" would replace velocity periodicity with the weak identity. That gives a different, larger class of competing solutions. Chapter 6 treats that literal reading, and chapter 8 treats every reading of both sentences in every combination.

When was the erratum added? We do not know. The PDF that Clay hosts shows a creation date of 2006 and a modification date of February 2013. That suggests the erratum was in place by early 2013, but it is a clue, not a date.

The exact text

From page 6 of the official description:

> The further condition $p(x + e_j, t) = p(x, t)$ should be made explicit in Eqn (8).

> Eqn (10) should read:
>
> $$-\iint_{\mathbb{R}^3\times\mathbb{R}} u\cdot\partial\theta/\partial t\,dx\,dt-\Sigma_{ij}\iint_{\mathbb{R}^3\times\mathbb{R}} u_i u_j\,\partial\theta_i/\partial x_j\,dx\,dt = \nu\iint_{\mathbb{R}^3\times\mathbb{R}} u\cdot\triangle\theta\,dx\,dt+\iint_{\mathbb{R}^3\times\mathbb{R}} f\cdot\theta\,dx\,dt+\iint_{\mathbb{R}^3\times\mathbb{R}} p\cdot(\operatorname{div}\theta)\,dx\,dt$$

Yesterday, today, tomorrow

Yesterday. The official description that Clay hosts ends with an Errata page written under Fefferman's problem statement. It asks that the pressure repeat from tile to tile in the periodic box, and it prints a corrected form of an integral identity.

Today. This chapter gives the exact text of the erratum and reads it in the most natural way: keep the velocity rule (10), correct the signs in the identity (12), and require periodic pressure of the solutions. This book calls that the erratum reading. The label "Eqn (10) should read" can also be taken word for word, which gives a larger class of competing flows; chapters 6 and 8 treat that. The date the erratum was added is not established by the sources we have, so the chapter says so.

Tomorrow. The two wordings are set side by side so that Clay, or any reader, can see exactly where they part. Which wording governs is not claimed here; that is for Clay to say.

For specialists

What the identity is. The printed (12) on page 3 is

$$\iint u\cdot\partial\theta/\partial t-\Sigma_{ij}\iint u_i u_j\,\partial\theta_i/\partial x_j=\nu\iint u\cdot\Delta\theta+\iint f\cdot\theta-\iint p\cdot(\operatorname{div}\theta).$$

The erratum's identity has the same convection, viscosity and force terms, with the time term and the pressure term reversed in sign, as integration by parts of the momentum equation (1) requires. It is therefore a correction of (12), not of (10).

What the pressure clause changes. Under the printed (10) and (11), the pressure of a smooth periodic solution has the form (periodic part) + $x \cdot a(t)$ + $C(t)$ (Tao 2011, Lemma 4.1(ii)). For a fixed flow u and force f, the pressure is unique up to $C(t)$, since two pressures have the same gradient. A nonzero tilt $x \cdot a(t)$ is not a free choice: changing it changes the flow, as in the sliding solutions of chapter 4. Requiring $p(x + e_j, t) = p(x, t)$ rules out every solution with a nonzero tilt. The added $C(t)$ remains harmless. Under the erratum reading, the solution class in (B) and (D) is (1), (2), (3), (10), (11) plus periodic pressure for all $t \geq 0$.

Which version governs the prize. The Millennium Prize rules (2018) say that a proposed solution must answer "the questions raised by the Problem's official description, as determined in the sole discretion of CMI" (§4(d)), and that only "a complete mathematical solution to a Problem as it is defined in its official Problem description" is eligible (§5(a)). The rules say nothing about errata. The official description Clay hosts today carries the errata page. This book makes no claim about the prize; it only sets the two wordings side by side.

Check it yourself

The exact text of the erratum is quoted in this chapter and in the exact statements page; compare it with the description hosted by the Clay Mathematics Institute.

Where to go next

- What did OpenAI answer under the erratum, and did our check agree? See OpenAI, 2026.
- What if the erratum's label is taken word for word? See What remains open.

5 OpenAI, 2026: what it answered

OpenAI, 2026: what it answered

The question

What did OpenAI's 2026 construction answer, and under which wording?

Why it matters

It is the published forced breakdown in the periodic box, and the wording of the problem decides what it answers.

What answering it would help with

Stating exactly what it answers, and under which wording, lets readers weigh claims about the prize problem.

What our research shows

In plain words

In 2026 the OpenAI team published Finite time blowup for Navier-Stokes, a 166-page paper, together with a Lean formalization, the package NavierStokesAndEuler (Apache License 2.0). Their construction builds a smooth force that drives a fluid, starting at rest, to a breakdown: its speed becomes unbounded as time approaches 1. They credit Diego Córdoba and Luis Martínez-Zoroa for the multiscale forced-singularity strategy behind it.

What OpenAI answered

- (D), the repeating box, under the erratum. Answered: yes, from rest. For every viscosity there is an allowed force for which no smooth flow from rest, with velocity and pressure both repeating, lasts forever. Checked in Lean, nanoda and lean4lean.
- (C), all of space. Answered: yes, by OpenAI's whole-space construction. For every viscosity there is a decaying starting flow and a decaying

force for which no smooth solution with bounded energy exists for all time. Checked in Lean and nanoda.
- So OpenAI's work answers (C), and (D) under the erratum as intended. The other two, (A) and (B), are the classical questions and stay open (chapter 5).

The repeating box

The construction is built on all of space and then repeated tile by tile to give the periodic version. In the periodic box, their pressure repeats too, and they say so directly. From page 125 of their paper:

> The pressure is periodic as well, as required in the erratum to the problem statement [13].

and, just before Corollary 10.6:

> The following corollary gives the periodic breakdown alternative (D) in [13].

Reference [13] is Fefferman's official description. So the OpenAI team aimed at (D) as the erratum states it, with periodic pressure. Under that reading, which keeps velocity periodicity (10), their construction meets it: there is no smooth solution, periodic in velocity and in pressure, that carries on past the breakdown time.

This is a large piece of work. The part of their Lean formalization needed for the periodic result alone runs to about 370,000 lines, and it builds and checks (see the check guide).

Where the printed wording comes in

The construction rules out every competing solution whose velocity and pressure both repeat. That is what the erratum asks under the reading that keeps (10). Under the original wording, the pressure need not repeat, so more competitors are allowed, including the sliding solutions of chapter 4. For a flow that starts at rest, one of those always exists (the slide-through escape:

starting from rest, for every allowed force, a smooth flow slides through the box forever with a tilting pressure). My idea: starting from rest, Terence Tao's 2011 slide (Proposition 1.7; for all time, sketched in Remark 6.2 with the details omitted) needs no assumption, because the resting fluid is already an unforced solution, so it applies to every push in Clay's class, OpenAI's included. I wrote a complete proof, for all time, of the case from rest, and checked every step but one in Lean (nanoda, lean4lean); its H^3 bound is stated in Lean, but without the clause that its series converges; that step is checked in writing. So, under the original wording, no breakdown from rest answers (D), for any force; for other starts, (D) is open: on every finite time interval, (D) holds exactly when (B) fails, by Tao's Proposition 1.7 (2011); for all time this holds only if the version his Remark 6.2 sketches, with the details omitted, holds; cited, not rechecked in Lean. It is where the two main wordings give different answers for (D). The answer key shows it cell by cell. Chapter 6 treats the reading in which the velocity need not repeat but the pressure must; it is open. Chapter 8 covers every other reading.

This chapter concerns the wording, not OpenAI's mathematics. The OpenAI team stated the condition they used and where it comes from.

Yesterday, today, tomorrow

Yesterday. In 2026 the OpenAI team published the construction above, with the pressure repeating too, as they say on page 125 of their paper.

Today. I ported the package to Lean 4.34.1 (533 of the 539 files used are byte-identical to the original commit; six carry a notice of the port) and rebuilt it with only the standard axioms. The Lean statement of the forced breakdown with repeating pressure says that for every viscosity there is an admissible force for which no smooth solution from rest, periodic in velocity and pressure, lasts forever; three further Lean statements restate it over broader force classes (every smooth repeating force, the Formal Conjectures force class, and every repeating starting flow). The forced breakdown with repeating pressure and the three wider statements were also accepted by nanoda and lean4lean. The whole-space breakdown for (C) was accepted by

nanoda (an export of 94,908 declarations). The mathematics of the construction is OpenAI's.

Tomorrow. The question of how closely the written paper and the Lean version agree step by step is a separate one that readers may take up, as chapter 7 describes.

For specialists

Corollary 10.6 (OpenAI, p. 125). "For every $\nu > 0$, there is a smooth force on $T^3 \times [0, \infty)$, compactly supported in time, for which the solution (U, P_per) of the periodic Navier-Stokes equations with viscosity ν and zero initial velocity is smooth on $[0, 1)$ and satisfies lim sup_{t↑1} ‖U(t)‖_{Lˆ∞(T³)} = ∞." Its proof ends with a uniqueness step: for two smooth periodic solutions with the same force and datum, "Periodic integration removes both the transport and pressure terms." Removing the pressure term this way uses periodicity of the pressure.

What OpenAI states publicly. The repository README describes the results as alternatives (C) and (D) "in the Clay Mathematics Institute's official problem description". This chapter's check of the periodic statement is the forced breakdown with repeating pressure. The whole-space statement (C) was checked separately, as the whole-space breakdown (Lean, with a nanoda export of 94,908 declarations). The clause-by-clause comparison of the whole-space statement with Clay's printed (C) is in the comparison note.

The Lean statement used here. The forced breakdown with repeating pressure: for every $\nu > 0$ there is a force f satisfying the printed (8)-(9) conditions such that no (u, p) exists with u smooth, p smooth, the momentum equation and incompressibility on $R^3 \times [0, \infty)$, u(x, 0) = 0, u periodic, and p periodic. The exact text is in the exact statements page. Three corollaries restate the conclusion as the failure of universal existence over broader force classes, in the comparator vocabulary of the Formal Conjectures project; the force class of the statement about every smooth repeating force omits the decay condition (9), so the forced breakdown with repeating pressure is the statement shaped like (D).

Provenance. The Lean sources here are OpenAI's package at commit f9e8bc5b38b6e212696e8a30e3e91517af887bbd. Of the 539 files used, 533 are byte-identical to that commit. Six carry a modified-file notice: for the port to Lean 4.34.1, three files take a Mathlib rename, one drops a redundant line, and one module is extracted into its own file and imported by another. The comparator definitions are © 2026 The Formal Conjectures Authors, Apache-2.0, and carry their original header. See the notice file.

Check it yourself

the check guide gives the commands to rebuild the ported package, print its axioms and check the nanoda exports.

Where to go next

- Does a Lean-accepted theorem match the written paper step by step? See A fidelity question raised by others.
- What happens under the wording as originally printed? See The original 2000 wording: what Tao saw in 2011, and what follows.

6 The original 2000 wording: what Tao saw in 2011, and what follows

The original 2000 wording: what Tao saw in 2011, and what follows

The question

What did Terence Tao observe in 2011 about the pressure in the printed periodic problem, and what follows from it under Clay's 2000 wording?

Why it matters

One sentence about pressure decides whether (D) from rest can be answered at all under the printed wording.

What answering it would help with

A formal check of the slide against the printed clauses settles which wording a claimed answer to (D) uses.

What our research shows

In plain words

Under the original 2000 wording:

- (D), starting from rest: no breakdown for any allowed force (Sonny, 2026; Lean). For every allowed force, a smooth flow from rest slides through the box forever with a tilting pressure, proved in writing, with every step but one also checked in Lean (and by nanoda and lean4lean); that the series in its H^3 bound converges is checked in writing. So no breakdown from rest, OpenAI's included, answers (D) under this wording. Printed (D) asks the solution only for (1), (2), (3), (10) and (11): the equations, the start, a repeating velocity and smoothness, with no energy bound. The slide meets these, and in the Lean statement its speed and the energy in each cell also stay bounded for all time. There is no contradiction with OpenAI's result: same push and same start, but a different solution, whose pressure does not repeat. For other starts, on every finite time interval, (D) holds exactly when (B) fails, by Tao's Proposition 1.7 (2011); for all time this holds only if the version his Remark 6.2 sketches, with the details omitted, holds; cited, not rechecked in Lean.
- (C) is answered: yes by OpenAI's whole-space construction (2026), which meets the ordinary printed (C), checked in Lean and nanoda.

The rest of this chapter explains the point about (D), which goes back to Terence Tao.

In August 2011 Terence Tao posted a long paper on the Navier-Stokes global regularity problem (arXiv:1108.1165; published in Analysis & PDE in 2013). Working from the problem as Fefferman stated it, he noticed that the

periodic problem asks the velocity to repeat but does not ask the pressure to repeat.

He pointed out what that allows. If the pressure may tilt steadily across space, the whole fluid can be made to slide through the box. His argument (Section 6 of the paper) has three steps. Step one: a pressure that tilts linearly across space takes up the force's average. Step two: the whole fluid is boosted in a direction of irrational slope, with a speed that grows in time; seen from the moving frame, the rest of the force oscillates faster and faster and tends weakly to zero (his Lemma 6.1). Step three: a stability result (his Proposition 5.3) turns a solution of the unforced problem into a smooth solution of the forced problem close to it. In general that step needs an unforced solution, which is Conjecture 1.4; from rest, the unforced solution is the resting fluid, so nothing has to be assumed. The pressure tilt does two jobs: it takes up the force's average, and it drives the acceleration of the moving frame. The Lean proof in this book follows the same plan by an explicit route, as its statement shows: the drift speeds up from rest to one fixed speed, set by the size of the force over all time, in the direction $(1, 2^{(1/3)}, 2^{(2/3)})$, and then holds it. The written proof bounds the flow relative to the moving frame explicitly in the Sobolev norm H^3, in place of Tao's compactness argument. The Lean statement carries that bound too, but without the clause that the infinite sum defining the norm converges (Mathlib gives a sum that does not converge the value 0); that step is checked in writing. Tao used it to show that, in the periodic box, the forced version of the "flow carries on" question is equivalent to the unforced one. What is the catch? The catch is the wording: the tilted pressure does not repeat, which the 2000 text allows and the erratum forbids. Tao called it "the technical loophole of non-periodic pressure".

He then proposed a repair: require the pressure to be normalised, which, up to an added constant, is the same as requiring it to repeat with the velocity. With that requirement the loophole closes, and he posed a "repaired" conjecture in those terms.

So by 2011 the pressure point was on the table, together with a fix. Whether Clay's erratum was already in place then, or was added afterwards, is not established by the sources we have (see chapter 2). We do not suggest that either one prompted the other.

Where the idea had appeared before

While tracing where this idea had appeared before, we found a comment left on Tao's blog on 10 September 2026 by an anonymous reader, who had noticed a closely related point about the pressure. It was posted under Tao's blog post of 7 September 2026 (on finite-time blowup with smooth forcing for related equations), and reads in full:

> In order to prove uniqueness in the T^3 case, the OpenAI solution assumes periodic pressure. Tao's argument shows that global unforced regularity on T^3 is equivalent to the existence of smooth solutions for forced data, allowing nonperiodic pressure. So the OpenAI solution does not imply the unforced result

The comment concerns the unforced question (B): it says that OpenAI's result does not settle it. It does not discuss whether OpenAI's push answers (D). We are grateful for that reader's curiosity. The internet works best as a place where people share what they notice, test it, and, with the right resources, work on it together. If you have a question you are curious about, or an idea you would like tested, please get in touch.

Yesterday, today, tomorrow

Yesterday. In 2011 Terence Tao observed that the printed periodic problem asks the velocity to repeat but not the pressure. He pointed out that a pressure tilting steadily across space lets the whole fluid slide, used this in Proposition 1.7 to show that the forced and unforced periodic questions are equivalent, and proposed a repaired conjecture with normalised pressure (Conjecture 1.8). On 10 September 2026 an anonymous reader of his blog connected that observation to the periodic pressure used in OpenAI's construction.

Today. I proved three results against Clay's printed clauses, in Lean. Two Lean results show that with no force and a fluid at rest, the printed periodic class allows both the resting solution and a uniformly sliding one (the conditional zero-force drift, and the bounded drift whose speed and cell energy stay below one). The slide-through escape shows that for every admissible force there is a smooth sliding solution from rest with a tilted pressure. My idea: starting from rest, Terence Tao's 2011 slide (Proposition 1.7; for all time, sketched in Remark 6.2 with the details omitted) needs no assumption, because the resting fluid is already an unforced solution, so it applies to every push in Clay's class, OpenAI's included. Sonny wrote a complete proof, for all time, of the case from rest, and checked every step but one in Lean (nanoda, lean4lean); its H^3 bound is stated in Lean, but without the clause that its series converges; that step is checked in writing. The two drift results are formalised in Lean by Sonny; the non-uniqueness is Tao's remark.

Today: Tao's 2011 question, Conjecture 1.8, now has an answer: no. I answered it, and checked it in Lean for every period. Tao asked in 2011 (Conjecture 1.8) whether every smooth periodic start and force have a smooth periodic solution with normalised pressure. I showed, and proved in Lean for every period, that OpenAI's Corollary 10.6 answers Tao's Conjecture 1.8 (the answer is no): by Tao's footnote 4, normalised pressure is periodic pressure up to a constant, and Corollary 10.6 rules out a smooth solution with periodic pressure. Two short written steps connect the Lean definitions to Tao's words. Start from rest and take OpenAI's force at viscosity 1, which is smooth and repeats with period 1. For every end time T of 1 or more there is no smooth periodic solution on $[0, T]$. This holds under each reading of Tao's normalised pressure (repeating; repeating with mean zero; his equation (9) coefficient by coefficient), and whatever period the solution has. It is checked in Lean in this book with only the standard axioms (nanoda also accepted it; the export sizes and the build limits are in the check guide), and it has not been refereed. The two written steps are: that Tao's smoothness on the closed slab $[0, T] \times R^3$ gives the smoothness the Lean statement uses is a standard fact; and that a pressure given by Tao's (9)

meets the coefficient-by-coefficient reading is a hand calculation, which the result does not need, because the weaker readings already reach Tao's statement. Both steps are written out in the Conjecture 1.8 written steps. The unforced case, Conjecture 1.4 (Clay's (B) on the torus), stays open.

Tomorrow. The formal statements may be reused by others who want to compare a wording of this problem with a proof. They may also serve as a template for checking other wording points in other problem descriptions.

For specialists

References are to arXiv:1108.1165v4 (31 May 2012); v1 (4 August 2011) numbers are in brackets.

- Section 1. "Following [15], however, we will not initially directly require any periodicity properties on the pressure." Reference [15] is Fefferman's official description.
- Proposition 1.7 [v1: 1.12], Elimination of forcing term. The forced periodic conjecture is equivalent to the unforced one. Tao: "Proposition 1.7 exploits the technical loophole of non-periodic pressure." It rests on the Galilean symmetry of the equations.
- Lemma 4.1(ii). The pressure of a smooth periodic solution is the normalised pressure plus an affine term $x \cdot a(t) + C(t)$.
- The repair. Normalised pressure $p = -\Delta^{-1}\partial_i\partial_j(u_i u_j) + \Delta^{-1}\nabla\cdot f$. Footnote 4 [v1: 7]: "Up to the harmless freedom to add a constant to p, this normalisation is equivalent to requiring that the pressure be periodic with the same period as the solution u."
- Conjecture 1.8 [v1: 1.13], Global regularity for periodic data with normalised pressure. "Let (u0, f, T) be a smooth periodic set of data. Then there exists a smooth periodic solution (u, p, u0, f, T) with the indicated data and with normalised pressure."

- Its refutation, and the period. Tao (p. 3) calls a solution periodic with period L when its data have period L. Data of period 1 also have every positive integer period N, so under Tao's definition the solution may have any of these periods. The Lean result covers every period $L > 0$,

which includes them, with the pressure asked only to repeat with period L; Tao's mean-zero and (9) readings at period L each imply that (a written step, as for period 1; see the Conjecture 1.8 written steps), so they are reached as well. For uniqueness at period L, the Lean proof rescales x to Lx and t to L^2t. Two solutions of period L with the same force and start become two solutions of period 1 with the same rescaled force, L^3 f(L^2t, Lx), and the same start. OpenAI's uniqueness theorem on the unit torus, which holds for any smooth periodic force with periodic pressure, then makes them equal. Shifting by a unit vector e_j gives a second solution with the same force and start, so the velocity has period 1. The equation then makes p(· + e_j) − p constant in space; a continuous function of period L is bounded, so the constant is zero. OpenAI's periodic corollary then excludes the solution. The Lean definitions follow Definition 1.1, the periodic clause, equation (9) and Conjecture 1.8 of v4, read against the paper.

Tao also remarks that the same loophole gives non-uniqueness for the periodic problem. The two Lean results on the drift in this book check that consequence against Clay's printed clauses: from a resting start with no force, the printed class allows both the resting solution and a uniformly sliding one. The slide-through escape checks a forced version of the slide (see the answer key and the Lean statements in the exact statements page). (Sonny, 2026; Lean.)

Check it yourself

The Lean statements are in the exact statements page, with the commands in the check guide. The two written steps of the Conjecture 1.8 result are in the Conjecture 1.8 written steps.

Where to go next

- What do the other alternatives look like under the erratum? See The other answers under the erratum.
- Which readings of the pressure sentence are there in all? See Every reading.

7 The other answers under the erratum

The other answers under the erratum

The question

Under the erratum, what are the answers for (A), (B) and (C), and for (D) read word for word?

Why it matters

The erratum changes the box alternatives, so each alternative has to be looked at again under it.

What answering it would help with

Having every alternative under the erratum in one place shows readers which cell a claim fills.

What our research shows

In plain words

Chapter 3 gave what OpenAI answered under the erratum: (D) is answered: yes from rest, and (C) is answered: yes on all of space. This chapter gives the other alternatives under the erratum, and the one wording that is still unsettled.

- (A), all of space, no push: open. This is the classical question. The erratum does not touch it, because the erratum names only equations (8) and (10), and (A) cites neither.
- (B), the repeating box, no push: open. This is also the classical question. The erratum's pressure rule does not change it: (B) is the same question under both wordings (Tao, 2011, p. 6 and Lemma 4.1); proved in Lean against Clay's clauses by Sonny. For every starting flow, with no force, a smooth solution with periodic velocity exists exactly when one with periodic velocity and periodic pressure exists. So adding the rule that the pressure repeats neither helps nor hurts (B).

- (C), all of space, push: answered: yes. The erratum does not touch (C) either, so OpenAI's whole-space construction meets it under the erratum within its stated equations.

- (D), the repeating box, push, with the erratum read word for word (the velocity need not repeat): from rest, every push has a smooth flow without a repeating-pressure requirement (Sonny, 2026), because the same slide-through escape from rest applies; and open with a repeating-pressure requirement. That open case is the subject of chapter 6: any escape for OpenAI's force would have to grow faster than linearly in space (Sonny, 2026).

Read as intended (the velocity and the pressure both repeat), (D) is answered: yes by OpenAI's construction from rest, as chapter 3 says.

Yesterday, today, tomorrow

Yesterday. The erratum asked that the pressure repeat in the periodic box, and printed a corrected identity under the label "Eqn (10)". It names only equations (8) and (10), which (A) and (C) do not cite.

Today. Under the erratum, (A) and (B) are the classical open questions, and (C) is answered: yes by OpenAI's whole-space construction. For (B), the pressure rule changes nothing: (B) is the same question under both wordings (Tao, 2011, p. 6 and Lemma 4.1); proved in Lean against Clay's clauses by Sonny. For (D) read word for word, the answer is no (answered: no) without repeating pressure, and open with it. The (A) to (C) cells under the erratum as intended are in the middle column of the answer key, and the cells for the erratum read word for word are in its right-hand column.

Tomorrow. The one open cell for (D), read word for word with repeating pressure, is stated exactly in chapter 6 so that others can attack it from either side. Further research is recommended on it.

For specialists

- (A) and (C) under the erratum. The erratum applies the pressure sentence and the corrected identity to equations (8) and (10) only. (A)

and (C) cite (4), (5), (6), (7) and never (8) or (10), so their wording and competitor class are those of the original. Applying the erratum to all of space is an additional reading, treated in chapter 8 (the pressure-on-all-of-space lemma, Sonny's (2026), makes (A) answered: no and (C) answered: yes under that reading; it is checked in Lean and nanoda for solutions smooth up to time zero and for solutions smooth for $t > 0$ with the pressure repeating at every positive time, and written for the rest). The clause-by-clause comparison for (C) is in the comparison note.

- (B) under the erratum. The Lean equivalence of printed and periodic-pressure existence is quoted in the exact statements page and used in chapter 8.

- (D) word for word. Without a periodic-pressure requirement the solution class contains the printed class, so the slide-through escape applies verbatim. With one, the status is that of the literal-erratum question of chapter 6.

Check it yourself

The equivalence for (B) and the slide for (D) are Lean statements in the exact statements page; the check guide gives the commands.

Where to go next

- What exactly is still undecided? See What remains open.
- How do these answers change under other readings of the text? See Every reading.

8 What remains open

What remains open

The question

When the erratum is read word for word with repeating pressure, what exactly remains open for (D), and what is known about it?

Why it matters

It is the one open cell for (D) under the erratum's wording, and its answer decides whether OpenAI's force meets (D) under that reading.

What answering it would help with

An exact statement, with the failed constructions listed, lets others attack it from either side.

What our research shows

In plain words

The main results. Under the 2000 wording, no breakdown from rest answers (D), for any push (Sonny, 2026; Lean). Under the correction as intended, OpenAI's push answers (D) from rest. This chapter is about a third reading, the correction read word for word, where one case is still open.

Three ways to read one sentence

Chapter 2 showed that the erratum can be read in more than one way. The sentence that matters here is "Eqn (10) should read:". Equation (10) is the rule that the flow repeats from tile to tile. Read word for word, the erratum would replace that rule with an equation that every smooth flow starting at rest already satisfies. In effect, the rule would simply disappear.

That gives three readings of alternative (D):

1. As printed. The flow must repeat from tile to tile. The pressure need not.
2. The erratum as intended. The flow must repeat, and so must the pressure. This is the reading the OpenAI team cite (p. 125), and their construction meets (D) under this reading (chapter 3).
3. The erratum read word for word. The flow no longer has to repeat. Whether the pressure has to repeat depends on whether one also keeps the erratum's other sentence, the one about pressure. So this reading splits in two: with repeating pressure, and without.

The words we need

- Periodic means "repeats like floor tiles". Space is cut into identical cubes, and a periodic flow looks the same in every cube.
- From rest means the fluid is perfectly still at the start, and only the force sets it moving.
- A competitor is any flow that obeys the equations, starts from rest, is pushed by the same force, and stays smooth forever. Alternative (D) asks for a force with no competitor at all. A single competitor is enough to defeat a force.
- Grows faster than linearly describes a flow whose speed, far from the starting cube, outruns every fixed multiple of the distance. However large a multiple you pick, somewhere far enough out the speed beats it. This need not happen in every direction; one direction is enough. A flow that repeats like tiles never does this: every tile looks the same, so its speed far away is the same as nearby.

The floor-tile picture

Picture a floor of identical tiles that goes on forever, with the same pattern painted on every tile. OpenAI's force paints its pattern on each tile in the same way. Their flow follows the same pattern on every tile, and it breaks down in every tile at the same moment, at time 1.

Under the reading "as intended", every competitor must be painted the same way on every tile, and OpenAI's work shows that no such competitor survives past time 1.

Under the word-for-word reading, a competitor may paint different tiles differently. So the question becomes: could a flow escape the breakdown by being different from tile to tile?

Why the pressure matters

Pressure is what pushes fluid from place to place. If the pressure may rise steadily from one side of the floor to the other, like a floor that tilts, the whole

fluid can slide across the tiles. Chapter 4 described that slide. A sliding flow from rest can always escape the breakdown, for any force (Sonny, 2026; Lean). So the word-for-word reading without repeating pressure is settled from rest: no breakdown from rest answers (D), for any force.

Under the word-for-word reading with repeating pressure, the floor cannot tilt, so the slide is gone. What follows is what is known for that case.

What is known, and what is not

- A small difference cannot help. We show that a competitor whose speed grows no faster than the distance must in fact repeat from tile to tile. It is then one of the flows that OpenAI already rules out.
- The difference must be large, and it must come early. Any competitor for OpenAI's force must grow faster than linearly somewhere, already before the breakdown time 1.
- Such wild flows exist. Even with no force at all, there are smooth flows from rest whose speed grows enormously far away. This is an old discovery about heat by Andrey Tychonoff (1935): with no limit on growth at infinity, "starting from nothing" does not force "staying at nothing". These flows exist because the floor is infinite, and they can draw on what happens far away.

- What is not known. Can such a wild flow carry OpenAI's force through the breakdown and stay smooth forever? If yes, then under this reading OpenAI's force does not meet (D). If no, it does. No known method settles this either way. We tried eleven explicit constructions; each one fails for a stated reason, listed below.

That is the open question this book leaves for OpenAI's force under this reading. Separately, Tao's 2011 question, his forced Conjecture 1.8, now has an answer, no, from Sonny's Lean proof (2026) using OpenAI's breakdown (chapter 4), while the unforced Conjecture 1.4, Clay's (B) on the torus, remains open.

Yesterday, today, tomorrow

Yesterday. Andrey Tychonoff showed in 1935 that for the heat equation, starting from nothing does not force staying at nothing unless growth far away is limited. The erratum can be read word for word so that the flow need not repeat from tile to tile, and then competing flows that grow wildly far away become possible in principle. OpenAI's Corollary 10.6 supplies the breakdown construction against which such flows are measured.

Today. I settled part of this reading and stated the rest exactly. Without a periodic-pressure requirement, no breakdown from rest answers (D), for any force, because the sliding flow of the slide-through escape applies (Lean, nanoda, lean4lean). With periodic pressure, a competitor that grows no faster than linearly in space must repeat and is excluded by the forced breakdown with repeating pressure, and any competitor for OpenAI's force must grow faster than linearly before time 1. Both statements are checked in Lean. Whether such a wild competitor exists is open, and eleven explicit constructions that were tried each fail for a stated reason.

Tomorrow. The open question is written so that others can attack it from either side: build a competing flow, or show that none exists. Narrowing it to superlinear growth tells a reader which tools to bring.

For specialists

Notation is that of the exact statements page. Names of results in this chapter are plain descriptions of what each result says; the Lean names behind them are collected in one code block at the end of the specialists' section. The Lean names are internal labels, not terms of the problem. Throughout, $\nu > 0$, f is smooth and 1-periodic in space, and (u, p) is a smooth pair on $\mathrm{R}^3 \times [0, \infty)$. "From rest" means $u(\cdot, 0) = 0$.

The three readings of (D)

Reading	Solution class (from rest)
(i) Printed	(1), (2), (3), (10), (11): u periodic; p unrestricted
(ii) Erratum as intended	(i) plus $p(x + e_j, t) = p(x, t)$
(iii-a) Erratum read literally, with periodic pressure	(1), (2), (3), (11) and the weak identity in place of (10), plus periodic p
(iii-b) Erratum read literally, without periodic pressure	(1), (2), (3), (11) and the weak identity in place of (10)

The weak identity adds nothing (checked by reading). The official text (page 3) takes the test fields θ smooth and compactly supported in $R^3 \times (0, \infty)$, so they vanish near $t = 0$. Integration by parts of (1) against such a θ gives the erratum's identity with no boundary term, for every smooth solution and from any start. No extension of u, p or f to negative times is needed. So class (iii-a) is exactly every clause of the printed solution class except velocity periodicity, plus periodic pressure. A Lean bridge (named in the code block at the end of this section) equates the smooth PDE competitor class with the comparator class together with periodic pressure. Its statement and proof contain no weak identity or zero-extension convention. The weak-identity equivalence is the smooth-solutions identity of chapter 8: written, and checked in Lean and nanoda for solutions smooth up to $t = 0$. The erratum's sign pattern was also checked numerically against an exact smooth solution: with the erratum's signs the pointwise defect is at round-off level, and with the printed signs of (12) it is of order one.

What is proved under each reading

Checker key: Lean = Lean 4.34.1, axioms propext, Classical.choice, Quot.sound only, no unproved placeholder; nanoda = accepted by that separate kernel checker on an export; lean4lean = replayed by that separate kernel checker as part of the whole environment that contains it.

(i) Printed. From rest, no breakdown answers (D), for any force; for other starts, (D) is open: on every finite time interval, (D) holds exactly when (B) fails, by Tao's Proposition 1.7 (2011); for all time this holds only if the version his Remark 6.2 sketches, with the details omitted, holds; cited, not

rechecked in Lean. The slide-through escape (proved in writing, with every step but one also checked in Lean, nanoda and lean4lean; that the series in its H^3 bound converges is checked in writing): for every admissible force, a sliding solution from rest exists forever, with pressure tilt $\langle$the force's average $-$ Y'', x$\rangle$. The conditional zero-force drift and the bounded drift (Lean, nanoda, lean4lean): two solutions from rest at zero force. For other starts, see chapter 8.

(ii) Erratum as intended. (D) is answered: yes by OpenAI's construction from rest. The forced breakdown with repeating pressure (Lean, nanoda, lean4lean); its three wider-class statements (Lean, nanoda, lean4lean). The escape-pressure criterion and the failure of the escape under repeating pressure (Lean, nanoda, lean4lean): the slide survives a periodic-pressure requirement only if Y'' = the force's average, and the universal escape is false under that requirement.

(iii-b) Literal, without periodic pressure. From rest, no breakdown answers (D), for any force. The class contains class (i), so the slide-through escape applies verbatim.

(iii-a) Literal, with periodic pressure. Open for OpenAI's force, narrowed as follows.

Result	Statement	Status
Shift lemma	If (u, p) is in class (iii-a), so is (u(· + e_j), p), with the same p and f.	Lean, nanoda
Period-defect system	The period defect $\delta = u(\cdot + e_j) - u$ solves $\partial_t \delta + (u(\cdot + e_j)\cdot\nabla)\delta + (\delta\cdot\nabla)u - \nu\Delta\delta = 0$, div $\delta = 0$, $\delta(\cdot, 0) = 0$. There is no pressure term, because both pressures are the same function.	Lean, nanoda
Nonzero-defect lemma	If f has no competitor in class (ii) (the forced breakdown with repeating pressure supplies such an f, with exactly this hypothesis), then every class-(iii-a) competitor for f has some j with $\delta \not\equiv 0$ solving the period-defect system.	Lean, nanoda
Subclass-exclusion lemma	If every literal-class competitor in a chosen subclass has zero period defects, and the force is excluded in the periodic class, that subclass has no competitor. The zero-defect hypothesis must be established separately.	Lean (standard axioms only, re-checked from copied source); the zero-defect hypothesis must be established separately
Linear-growth repetition theorem	At most linear growth forces periodicity. If (u, p) is in class (iii-a) and $\lvert u(x,t)\rvert \le M(1 + \lvert x\rvert)$ on $R^3 \times [0, T]$, then $u(\cdot + e_j, t) = u(\cdot, t)$ on $[0, T]$ for each j. Proof: weighted energy for δ with weight $\exp(-\lambda(t)(1 + \lvert x\rvert^2))$, $\lambda(t) = e\hat{}\{K(t-t_0)\}$, $K = 24M + 4M^2/\nu + 32\nu$, cutoff $\chi(x/R)^2$, on steps of length $(\ln 2)/K$. The stretching term is integrated by parts using div $\delta = 0$, so only the size of u enters, not its gradient. This is the classical weighted-energy uniqueness argument for parabolic problems in a growth class (Tychonoff).	Lean, nanoda. The Lean proof takes the same route with different constants: weight $\exp(-\lambda(t)(1 + \lvert x\rvert^2))$ with $\lambda(t) = 1 + K(t - t_0)$, steps of length $1/K$, $K = 24M + 8M^2/\nu + 96\nu$. It uses only that the force repeats
Superlinear-growth theorem	For OpenAI's force, growth must be superlinear before $t = 1$. Every class-(iii-a) competitor u for OpenAI's force satisfies sup over $R^3 \times [0, T]$ of $\lvert u(x,t)\rvert/(1 + \lvert x\rvert) = \infty$ for some $T < 1$. Proof: otherwise the linear-growth repetition theorem makes u periodic on every $[0, T]$, $T < 1$; At viscosity one, OpenAI's periodic uniqueness theorem on a closed slab gives $u = U$ on each closed slab before time 1; an application at another viscosity additionally needs the appropriate viscosity-scaling bridge; continuity of u on the compact set $[0,1]^3 \times [0,1]$ then contradicts lim sup_{t↑1} ‖U(t)‖_∞ = ∞ (OpenAI, Corollary 10.6, p. 125).	Lean, nanoda. Corollary 10.6 enters as OpenAI's own Lean theorem, and the viscosity-scaling bridge is Lean
Fast-growing members	The class has fast-growing members. At zero force, $u = (H(t, x_2), 0, 0)$, $p = 0$, with $H(t, y) = \Sigma_n\, g^{(n)}(t)\, y^{2n} / (\nu^n\, (2n)!)$, $g(t) = \exp(-1/t^2)$ for $t > 0$ and 0 for $t \le 0$, is a smooth, nonzero, non-periodic class-(iii-a) solution from rest. Being non-periodic, it grows faster than linearly in x_2, by the linear-growth repetition theorem. The bound $\exp(c\, {x_2}^2/t)$ of the written argument is an upper bound, not a proved growth rate.	Written (complete argument: the heat-shear control in the literal-erratum proofs, section 5); not in Lean
Time-slab non-	Uniqueness already fails on a time slab for OpenAI's force. OpenAI's force and solution vanish on $[0, t_0]$, $t_0 = 1 - \ell^{-2} \in (0, 1)$, where $\ell > 1$ is	Written (the positive initial interval on which the force is

Result	Statement	Status
uniqueness	a parameter of their construction. On that slab the solution of the fast-growing members result is a competitor to U = 0.	zero is Lean; the fast-growing members result is written)

The sources of the four literal-erratum lemmas are a single Lean file, built against this book's own statement and geometry files. That file is now in this book's Lean folder for the literal-erratum lemmas, in a separate build target. A Lean statement joining the nonzero-defect lemma directly to the forced breakdown with repeating pressure and to the comparator class of unrestricted-velocity competitors is the bridge named in the code block below. It has been checked by Lean and nanoda, and it was rebuilt from source (standard axioms only; the replay tool leanchecker passed).

Computed. The erratum's sign pattern (see above). The defect identity for additive shear competitors $u = U + H(t, \langle k, x\rangle)d$, and the linear-plus-periodic identity $(A' + A^2)e_j + (Ae_j\cdot\nabla)U = 0$ behind rows 1 and 3 of the table below, were checked to round-off (relative error of order 10^{-14}) with analytic derivatives.

The open question

> Question. Fix $\nu > 0$. Let f be the force of OpenAI's construction for this ν, the force behind Corollary 10.6 and the forced breakdown with repeating pressure. Is there a pair (u, p) in $C^\infty(R^3 \times [0, \infty))$ with
>
> $\partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = f$, div $u = 0$, $u(\cdot, 0) = 0$, $p(x + e_j, t) = p(x, t)$ for $j = 1, 2, 3$,
>
> and no periodicity or growth condition on u?

- By the superlinear-growth theorem, any such u has sup over $R^3 \times [0, T]$ of $|u|/(1 + |x|) = \infty$ for some $T < 1$. "Superlinear" means exactly this: the ratio to $1 + |x|$ is unbounded on a slab. It is not a pointwise growth rate.
- By the nonzero-defect lemma, any such u carries a nonzero period defect solving the pressure-free period-defect system from zero data.

- YES means that, under reading (iii-a), OpenAI's force does not meet (D) from rest.
- NO means that it does.
- Other forces and other starting flows are outside this question. Under reading (iii-a), (D) asks only that some force and start work.

Eleven constructions that fail

Each is an attempt at a YES. Each fails for the reason given.

#	Construction	Why it fails
1	$U + W$, $W = (H(t, x_2), 0, 0)$ a heat shear	Cross terms $U_2 \partial_2 H\, e_1 + H \partial_1 U$ do not vanish for OpenAI's U, are not periodic in x_2, and cannot be absorbed by a periodic pressure. And $U + W$ still contains U, which breaks down at $t = 1$. Excluded, with every other competitor of this form, by the corollary below (Lean, nanoda).
2	A shear independent of the variables U depends on	Needs $\partial_1 U = 0$ and $U_2 = 0$: a 2½-dimensional, globally regular field, not OpenAI's U.
3	$u = A(t)x +$ periodic part	Periodic p forces $A' + A^2 = 0$; from rest, $A \equiv 0$. Linear growth is also excluded by the linear-growth repetition theorem.
4	Couette shear $u = S\, x_3\, e_1$	Not at rest unless $S = 0$; ramping $S(t)$ makes A' non-symmetric, so it is not a pressure gradient.
5	Galilean drift (Tao's slide, the slide-through escape)	Pressure tilt $\langle$the force's average $- Y'', x\rangle$ is periodic only if Y'' = the force's average (the escape-pressure criterion); the failure of the escape under repeating pressure refutes the periodic-pressure escape. The velocity stays periodic anyway.
6	Rotating frame	Rotates the force, so f is not kept; centrifugal term gives a quadratic, non-periodic pressure.
7	Scaling $\lambda u(\lambda x, \lambda^2 t)$	Changes the force to $\lambda^3 f(\lambda x, \lambda^2 t)$.
8	Velocity periodic on a coarser lattice NZ^3	Bounded on finite slabs, so the linear-growth repetition theorem makes it 1-periodic.
9	A heat shear carrying the force alone	Shears solve NS only for forces $(F(t, x_2), 0, 0) + \nabla$(periodic); OpenAI's force is three-dimensional.
10	Heat shear switched on later, $H(t - t_1, x_2)$	Same cross terms as #1 after t_1; equal to U before. Excluded, with every other competitor of this form, by the corollary below (Lean, nanoda).
11	Drift driven by viscosity instead of a pressure tilt (heat series with a time-dependent axis value)	Grows faster than any Gaussian in x_2; the periodic part then meets a strong, x_2-dependent shear and a global solution on R^3 with such coefficients is needed. No closed form exists; this is the open question itself.

The additive shear is closed for OpenAI's force. I proved the strong form for OpenAI's U, and from it that no additive-shear competitor exists, for any k and d. The setting: U is OpenAI's periodic solution, as ported in this book, and a competitor u has the same force, a repeating pressure, and the form $u = U + H(t, \langle k, x\rangle)d$ with $\langle k, d\rangle = 0$. Strong form. For $0 < t < 1$ and $d \neq 0$, $U(t, x) = 0$ wherever some period difference of $H(t, \cdot)$ is nonzero at $\langle k, x\rangle$. The proof uses the structure of OpenAI's U. Its support stays strictly inside the unit cell, so $U(t, \cdot)$ vanishes on every half-integer coordinate plane, and every

line in direction d meets a zero of U. Where a period difference is nonzero, subtracting the two momentum equations makes $(d\cdot\nabla)U$ parallel to d. That set is open and invariant along d. On it, the part of U orthogonal to d is constant along each line and vanishes at that zero; then incompressibility makes the remaining part vanish too. Corollary. There is no global smooth competitor of this form, for any k and d (k = 0 and d = 0 included) and any function H. The proof: u is bounded on $[0, 1] \times [0, 1]^3$, say by M, while OpenAI's U has a point with $|U(t, x)| > 2M + 1$ before $t = 1$. U is nonzero on the whole orbit $x + Z^3$, so by the strong form $H(t, \langle k, \cdot\rangle)$ repeats along that orbit, and $|u(t, x)| \leq M$. The zero of U on the d-line through x has the same value of $\langle k, \cdot\rangle$, which gives $|H(t, \langle k, x\rangle)d| \leq M$. So $|U(t, x)| \leq 2M$, a contradiction. The corollary needs only that u is smooth for $0 < t < 1$ and continuous for $0 \leq t \leq 1$, with the shear form for $0 < t < 1$. It uses no growth bound, no start from rest and no divergence condition on u. It excludes constructions 1 and 10 above, and every other competitor of this form. Proved in Lean and nanoda (standard axioms only) for the chapter's own class: the force of the forced breakdown with repeating pressure and the unrestricted-velocity competitors named in the code block below. Written proof in the additive-shear proofs. The question above is unchanged. A competitor that is not of this form is neither built nor excluded. The strong form fails for some other periodic solutions, so any proof of it has to use the structure of OpenAI's U. Counterexample: with force e_3, the periodic solution $U = t\, e_3$ (pressure 0) and the heat shear H of the fast-growing members result, $u = t\, e_3 + H(t, x_2)e_1$ solves the equations from rest with pressure 0, while $U \neq 0$ at every point with $t > 0$ and $H(t, \cdot)$ has nonzero period differences at arbitrarily small times (a heat shear that repeated in x_2 would be bounded, hence zero). Proved in writing (a second reading within this project; the additive-shear proofs); checked numerically. What holds for every periodic U with periodic pressures: subtracting the two momentum equations and comparing x with x + e_j shows that, wherever a period difference of $H(t, \cdot)$ is nonzero at $\langle k, x\rangle$, the part of $(d\cdot\nabla)U(t, x)$ orthogonal to d is zero. This needs only that both solutions have the same force and repeating pressures; it uses neither the heat equation for H nor the start from

rest. Proved in writing (a second reading within this project; the additive-shear proofs); not in Lean.

Why it is research-level in both directions

- YES would need a global smooth Navier-Stokes flow, growing faster than linearly in space, that leaves U's path before $t = 1$ while keeping a periodic pressure. The freedom would have to come from spatial infinity, as in Tychonoff's examples. No method is known that builds such a solution for the nonlinear equations with a smooth force that is not analytic in time.
- NO would need uniqueness, or a breakdown, for every such fast-growing branch. The time-slab non-uniqueness result shows that uniqueness without growth bounds already fails on $[0, t_0]$ for OpenAI's own force, and the heat shear is nontrivial at every positive time (its value at $y = 0$ is positive, although exact enclosures at $t = 1/10$ show it changes sign between $y = 15/4$ and $y = 4$, so it is not nonzero at every point). So no argument local to the singularity can settle it, and the known uniqueness theorems all assume a growth class.

- Both directions are questions about the Navier-Stokes equations without growth conditions at infinity. They concern the literal reading only. Readings (i), (ii) and (iii-b) are settled from rest, as above.

Credit

We stand on the shoulders of giants. Charles L. Fefferman wrote the problem statement; the Errata page is hosted by the Clay Mathematics Institute with it. The OpenAI team built the construction: their force, Corollary 10.6, the periodic uniqueness step and the NavierStokesAndEuler package (Apache-2.0). Their construction meets (D) under the erratum as intended, and they point to the erratum's requirement themselves ("The pressure is periodic as well, as required in the erratum", p. 125). The slide-through escape rests on a mechanism that is Terence Tao's (2011); applying it from rest to every push in Clay's class, the complete all-time proof and the Lean check are Sonny's

(2026). The sliding solution is also in Koch, Nadirashvili, Seregin and Šverák (2009), and the sliding-frame symmetry in Fushchych and Popovych (1994). A. N. Tychonoff (Théorèmes d'unicité pour l'équation de la chaleur, Mat. Sb. 42, 1935) gave heat-equation non-uniqueness without growth bounds, and uniqueness within growth classes. The kernel checkers are nanoda and lean4lean.

Below are the names behind the results of this chapter. The Lean names are internal labels, not terms of the problem.

```
-- printed solution class: Statement.PrintedSolution
-- erratum as intended: PrintedSolution v f u p ∧ Statement.Periodic p
-- erratum read literally, with periodic pressure: NSClose.LiteralSolution
--   (class (iii-b) is the same without the field Periodic p; other fields
-- bridge to the comparator class: NSCloseBridge.literal_iff_unrestricted
-- shift lemma:                NSClose.shift_literal
-- period-defect system:       NSClose.defect_system
-- nonzero-defect lemma:       NSClose.competitor_carries_nonzero_defect
-- subclass-exclusion lemma:   NSClose.subclass_exclusion
-- bridge joining the nonzero-defect lemma to the forced breakdown with rep
--   NSCloseBridge.openai_force_literal_competitors   (file lean/NSClose/Br
--   comparator class NSCloseBridge.VelocityUnrestricted (a verbatim copy,
-- files: lean/NSClose/Literal.lean (built against lean/JustCheckin/Stateme
-- build target: ReadingsClose
-- additive shear (file lean/AdditiveShear.lean; build target ReadingsAddit
--   strong form for OpenAI's U:            AdditiveShear.shear_forces_zero
--   no additive-shear competitor:          AdditiveShear.no_additive_shear
--                                          AdditiveShear.no_additive_shear
--   in the chapter's class, with the force of the forced breakdown with re
--                                          AdditiveShear.chapter6_no_addit
--                                          AdditiveShear.literal_no_additi
-- OpenAI's periodic uniqueness theorem on a closed slab: classical_uniquer
-- initial zero-force interval: lean/InitialGap.lean
-- linear-growth repetition theorem and superlinear-growth theorem (files l
--   OWLinear.lean and LinearGrowthCalc ... LinearGrowthMain.lean; build ta
--   OriginalWording.linear_growth_repetition : OriginalWording.LinearGrowt
--   OriginalWording.superlinear_growth
--   OriginalWording.superlinear_growth_of_linear_repetition   (the same, f
```

Check it yourself

The Lean results of this chapter are quoted in the exact statements page, with the commands in the check guide. The written parts are in the literal-erratum proofs.

Where to go next

- Which readings leave other alternatives open? See Every reading.
- Is the Lean statement the one in the English problem? See A fidelity question raised by others.

9 A fidelity question raised by others

A fidelity question raised by others

The question

Does the gap that others found between OpenAI's written paper and its Lean version affect anything in this book?

Why it matters

Lean acceptance shows that a Lean theorem holds. A reader still has to ask whether a written proof matches it.

What answering it would help with

Saying exactly what rests on Lean statements lets readers weigh each cell of the answer key.

What our research shows

In plain words

In 2026 A. Bastounis, F. Circelli and A. C. Hansen asked whether OpenAI's written paper and its Lean version agree step by step (arXiv:2610.08144); their point is about the gap between a paper written in ordinary

mathematical language and its Lean version. Lean checking a proof tells you the Lean theorem is true. It does not, by itself, tell you that every step of the written paper is right, because the Lean proof may reach the same theorem by a different route. They give examples from OpenAI's Navier-Stokes package where a lemma in the paper and its Lean counterpart differ, and they say plainly that they make no claim about whether OpenAI's written proof is correct.

Does it affect anything in this book? The answer key rests on Lean statements: what exactly each theorem says, how it lines up with Clay's clauses, and that it is accepted by Lean and by separate kernel checkers. Their paper agrees that Lean acceptance tells us the Lean theorem holds. So their challenge leaves those cells unchanged. Some results in this book are of other kinds, and each is labelled where it appears. The link from printed (D) at other starts to (B) is Tao's written argument. Of the fifteen lemmas of chapter 8, two are checked in Lean in full (the pressure-periodicity lemma and the curl of OpenAI's force) and thirteen are written proofs with parts checked in Lean (two of them only conditional on a named premise); the seven collections of Lean statements added in the second round, among them Clay's 2000 wording stated directly and Tao's Conjecture 1.8, are checked in Lean. The two growth theorems of chapter 6 are checked in Lean, and the superlinear-growth theorem uses OpenAI's Lean theorem for Corollary 10.6. Two of chapter 8's lemmas use OpenAI's Corollary 10.6, or their zero-force window, as stated, and so does the Conjecture 1.8 result of chapter 4. Those results inherit whatever holds for the statements they use.

This book reports exactly what the Lean statements say. Still open: the link between those formal statements and OpenAI's written paper.

Check it yourself

Each Lean statement this book relies on is quoted in the exact statements page, and the check guide gives the commands.

Where to go next

- Which verdicts rest on Lean statements, and which on written proofs? See Every reading.
- What did OpenAI actually prove? See OpenAI, 2026.

10 Every reading

Every reading

The question

Under each of the 440,401,920 readings of Clay's text, which of (A) to (D) is answered yes, answered no or open, and on what basis?

Why it matters

A verdict means little without the reading it is made under. Listing every reading leaves nothing to guess.

What answering it would help with

The four-question guide and the decision lists let later papers say which wording they use.

What our research shows

In plain words

Clay's official description of the Navier-Stokes problem can be read in more than one way. A few of its sentences carry more than one meaning, and the erratum adds two more. This chapter takes every reading of every ambiguous sentence that has been raised, in every combination, and says what follows for each of the four alternatives (A) to (D). There are different ways of reading the question, and this chapter gives the answer under each.

The order in which you apply the sentences does not matter. Each reading adds a condition on the starting data or on the solutions, and conditions

simply stack. So the answer depends only on which readings you choose. The one place where order is part of the wording, "take $\nu > 0$, then choose the data", is itself one of the clauses examined.

There are 440,401,920 combinations. That sounds like a lot, but most sentences do not change the answer once a few others are fixed. So the combinations fall into a short list of cases, and each case has one answer with the reason behind it.

Each answer is one of:

- Answered: yes: the alternative is proved true under that reading (by Lean, by a written argument given here, or by a classical theorem we cite).
- Answered: no: the alternative is proved false under that reading, so nobody can prove it.
- Open: neither is known yet. Each open case is stated exactly below.

A "no" is a proved result, just as a "yes" is.

For (D) there are two separate questions. Is (D) true under the reading? And does OpenAI's construction (their force, starting from rest) meet it? In the tables for OpenAI's construction, "answered: yes" means the construction meets the reading, and "answered: no" means it is proved not to; that does not by itself decide (D) for other forces. The two questions can differ: a reading can make (D) easy to meet with some other force while OpenAI's force has an escape.

Find your reading here

Answer four questions in order. Then read your row in the table.

1. Must the flow repeat from tile to tile? The printed rule (10) says yes. The erratum's label "Eqn (10) should read" could mean that (10) is replaced. If you read it as a slip for (12), the answer stays yes.
2. Must the pressure repeat? Never, or only at the starting moment, or at every moment.

3. Do you treat an integral identity as an extra rule? The erratum's corrected identity adds nothing, because every smooth solution satisfies it. Requiring the uncorrected printed (12), or an identity with only one of the two signs corrected, is a different matter (see the last rows below).
4. Do you add anything the text does not list? Examples are finite energy over all of space, decay in space for the force, zero average velocity, an energy inequality, limits on how fast the pressure grows, weak (non-smooth) solutions, or smoothness only after time zero.

If your answer to question 4 is "nothing", and to question 3 is "no extra identity":

Flow must repeat?	Pressure must repeat?	Is (D) true?	Does OpenAI's construction meet (D)?	(B)	Why
yes	never	from rest: every force has a smooth flow (Sonny, 2026); other starts: open; on every finite time interval, (D) holds exactly when (B) fails, by Tao's Proposition 1.7 (2011); for all time this holds only if the version his Remark 6.2 sketches, with the details omitted, holds; cited, not rechecked in Lean	no: a sliding flow escapes	Open	the slide-through escape (Lean, nanoda, lean4lean)
yes	at the start only	Open	no: a sliding flow, adjusted while the force is still zero, escapes	Open	tilt removal while the force is zero (written; OpenAI's zero window is Lean)
yes	always	yes	yes	Open	the forced breakdown with repeating pressure (Lean, nanoda, lean4lean)
no	never	from rest: every force has a smooth flow (Sonny, 2026); other starts: Open	no: the same sliding flow escapes	Open	the slide-through escape
no	at the start only	Open	no	Open	tilt removal while the force is zero
no	always	Open	Open: the chapter 6 question	Open	chapter 6

(A) is open in every one of these rows. (C) is answered: yes for the ordinary printed wording and for the erratum within its stated equations (the whole-space breakdown; section 4, the ordinary printed (C)).

If you answered question 3 with an extra identity (printed (12), or a half-corrected identity), (D) becomes true under every pressure and velocity reading, and (B) becomes false. Under the 2000 wording, with the identity imposed on (A) itself, the text's test fields and solutions smooth up to $t = 0$, (A) becomes false too (Lean, nanoda; section 4 lists the readings of this kind that stay open).

- OpenAI's force. With printed (12), or with only the pressure sign corrected, OpenAI's force meets (D). The reason has nothing to do with blowup: the extra identity forces a flow from rest to be a gradient flow, and OpenAI's force cannot drive one (the extra-identity reduction, the frozen-vorticity obstruction and the curl of OpenAI's force; section 5). With only the time sign corrected, the identity forces the pressure to be constant in space. OpenAI's force then meets (D) when the flow must repeat; when it need not, the question is still open.
- (B). Explicit starting flows have no solution at all (the shear-start obstruction).
- These readings settle (D) for reasons unrelated to the construction, so they lend it no support.

If you added something in question 4, look it up in section 4. The short version:

- Finite energy over all of space, or a smooth zero extension before time zero: (D) true; (B) false.
- Decay in space as in (4), (5): the only periodic data left are zero. (D) false; (B) true; OpenAI's force is excluded.
- Weak solutions: classical weak solutions always exist (Leray, Hopf). (D) false; (B) and (A) true. This changes if an extra identity is also imposed (section 4).
- Zero average velocity at every time: (D) true for a trivial reason. OpenAI's construction meets it with a repeating flow when the data are chosen for each ν or for one ν, whatever the pressure reading; it does not with one force for all ν and no $\nabla p = 0$ requirement; the rest (a free flow with data for each ν or one ν, or $\nabla p = 0$ with one force for all ν) is open (section 4).

- Smoothness only after time zero: the same answers as without it, apart from one corner that departs from the text twice ((A) with the pressure repeating at time zero only and smoothness only afterwards, which is answered: no).

- Pressure growing slower than linearly: with periodic flow, the same as "pressure always repeats".
- An energy inequality on a cell: OpenAI's construction meets it with a repeating flow (tilted pressure allowed) when the data are chosen for each ν or for one ν; it does not with one force for all ν and no $\nabla p = 0$ requirement; the rest (a free flow with data for each ν or one ν, or $\nabla p = 0$ with one force for all ν) is open (section 4).

Every one of these readings adds a condition the text does not contain. Section 2 names the sentence that each one departs from.

Yesterday, today, tomorrow

Yesterday. Clay's description has several sentences that can be read in more than one way, and the erratum adds two more. Leray (1934) and Hopf (1951) gave global weak solutions, Seeley (1964) proved a theorem on extending smooth functions from a half-space, and Tychonoff (1935) showed that uniqueness needs growth limits. In the sources we cite, including Tao's 2011 paper and OpenAI's paper, each work states the reading it uses.

Today. I took every reading of every ambiguous sentence that has been raised, in every combination: 440,401,920 combinations, which fall into a short list of cases. Each case has one answer, marked answered: yes, answered: no or open, with the reason beside it, and each answer says whether it rests on Lean, on a classical theorem or on a written proof (several written proofs have Lean helper statements). The counts below can be recounted from the decision lists in section 4 (the recipe is in the check guide), and the reading splits describe each split in plain words. Counted over all combinations, (A) is answered: no in 57.3%, open in 31.3% and answered: yes in 11.4%; (B) is answered: no in 72.0%, answered: yes in 16.4% and open in 11.6%; (C) is answered: yes in 65.9%, answered: no in 11.4% and open in 22.7%; and (D) as a statement is answered: yes in 75.8%, answered: no in 16.3% and open in 7.9% (rounded). The classical (A) question, the classical (B) question and the remaining readings of (D) are the questions that stay open as questions.

How much is answered. Each bar shows the share of the 440,401,920 readings that now have a proved answer (yes or no); the light part is still open. These shares count readings equally; they say nothing about which readings are reasonable.

Question	Answered	
(A) All of space, no push		68.7%
(B) Repeating box, no push		88.4%
(C) All of space, with a push		77.3%
(D) Repeating box, with a push		92.1%
All four		81.6%

Groups of readings decided by written proofs, with Lean helper statements, include (A) with the pressure repeating at time zero only and smoothness only afterwards; (D) and (B) with weak solutions and a remaining degenerate identity; and parts of (D) with a start of minimal period one, of OpenAI's construction with weak solutions and pressure freedom, of (A) and (C) with weak solutions and the erratum applied to all of space, of the zero-mean force, one-force-for-every-viscosity, zero-mean-velocity, cell-energy-inequality and one-cell-identity readings, of OpenAI's construction with weak solutions and zero mean velocity, and of the ordinary printed (C) and its variants (section 4 and the reading splits). (C) with weak solutions and the zero-extended identity is answered: yes. OpenAI's construction with weak solutions and zero mean velocity is answered: no. The one-cell-identity readings split three ways: zero-mean force answered: yes, one force for every viscosity answered: no, pressure gradient vanishing on the cell open. The ordinary printed (C) is answered: yes for the checked port of OpenAI's whole-space construction (the whole-space breakdown), and also for the usual energy inequality, for the energy bound with a pressure norm added, and for a force of zero integral. Three groups of (C) stay open: one common force for every viscosity (52,068,352 readings, 11.8% of all combinations); energy finite at each time only, or a divergence-free force, in the ordinary printed (C) (45,559,808); and weak solutions with the erratum applied to all of space and a cell identity or a pressure at time zero only (2,359,296). The (A) readings in

which printed (12), or an identity with only one sign corrected, is imposed on (A) itself while the erratum applies to the box only are answered: no on the text's test fields and with smoothness up to time zero, by a Lean result stated directly against Clay's 2000 wording (16,515,072 readings).

Tomorrow. The four-question guide lets any reader find their own row and see what follows. The open questions are listed exactly in section 6, and the written proofs marked for Lean checking may be turned into Lean statements by others.

For specialists

Notation follows the exact statements page. $\nu > 0$ throughout; "from rest" means $u(\cdot, 0) = 0$; f is OpenAI's force for the given ν (the force behind Corollary 10.6 and the forced breakdown with repeating pressure) unless a row says otherwise. A competitor for a force f is a pair (u, p) in the solution class of the reading, from rest, defined on $\mathrm{R}^3 \times [0, \infty)$.

Names of results in this chapter are plain descriptions of what each result says; they are not terms of the problem. The Lean names behind the Lean-checked parts are collected in one code block at the end of section 5.

1. What the official text settles

Source: C. L. Fefferman, Existence and smoothness of the Navier-Stokes equation, the PDF hosted by the Clay Mathematics Institute, with its Errata page (the copy read here has SHA-256 c1b5f27b1a64705cfaf1afceea513db5deedca8a18ca56ab32e7f86445a06d0c). The clauses quoted below are now in the exact statements page, Part 1.

- Test functions of (12). Page 3: "for any smooth vector field $\theta(x, t) = (\theta_i(x, t))_{1 \le i \le n}$ compactly supported in $\mathrm{R}^3 \times (0, \infty)$, a formal integration by parts yields (12)". Tests are arbitrary vector fields (not only divergence-free) and vanish near $t = 0$. So no initial-trace term, no extension of u, p, f to negative times, and no time-periodicity convention enters any identity.

- (12) is prose. It is introduced as a consequence of (1), (2) ("if (1) and (2) hold, then ... yields") in the paragraph on weak solutions. None of (A)-(D) lists (12) among its conditions. Only the erratum's label "Eqn (10) should read" brings an identity into a condition list.
- (B) and (D) cite (1), (2), (3), (8), (9), (10), (11) only. (9) is assumed "in place of (4) and (5)". (A) and (C) cite (4), (5), (6), (7) and never (8) or (10), the two equations the erratum names.

- (A) and (B) ask for "smooth functions p(x, t), u_i(x, t)", and (11), which no erratum reading touches, is C^∞. Weak-only readings depart from these words.
- (7) reads "∫_{R^n} |u(x, t)|² dx < C for all t ≥ 0 (bounded energy)": one constant for all t.
- (D) quantifier order. "Take ν > 0 and n = 3. Then there exist ...": data may depend on ν. The forced breakdown with repeating pressure has exactly this shape: for every ν > 0, there is an f.

2. The clauses and their readings

Sixteen clauses. "Text" marks the reading that matches the words; every other reading is examined anyway.

Clause	Readings	Departs from
The erratum's pressure sentence	not applied; a condition on data only; a free variable with no binder; at $t = 0$ only; modulo affine functions (∇p periodic); for all $t \geq 0$; for all $t \geq 0$ modulo $C(t)$	none of these is fixed by the text, which places a solution condition in the data list (8)
The erratum's label "Eqn (10) should read"	ignored; a slip for (12), (10) kept; identity added, (10) kept; (10) replaced by the identity (literal)	none fixed by the text
Which identity binds	only what the label says; printed (12) also binding; only the time sign corrected; only the pressure sign corrected	the last three: the text prints only the printed (12) as prose and the erratum's identity
Test functions	text (vector fields, compact in $R^3 \times (0, \infty)$); divergence-free only; reaching $t = 0$ with zero extension; spatially periodic on one cell	text, page 3
Smoothness (6), (11) at $t = 0$	C^∞ up to $t = 0$ (Lean's notion of smooth on the closed half-space); restriction of a C^∞ function on $R^3 \times R$; C^∞ for $t > 0$, continuous at $t = 0$; zero extension C^∞ across $t = 0$; weak solutions	"$C^\infty(R^n \times [0, \infty))$"; "smooth functions"
Where (1) holds	$t \geq 0$; $t > 0$	—
Force bounds (9)	K natural or real; Euclidean norm or components	—
Extra force conventions	none; divergence-free force; zero spatial mean; compact support in time	(8), (9) list no such condition
Periods in (8), (10)	unit shifts e_j; all integer shifts; minimal period exactly 1; solutions allowed period N (lattice NZ^3)	the equations in (8), (10)
Viscosity	data chosen for each ν; one ν; one force for all ν	"Take $\nu > 0$ . . . Then there exist"
Extra conditions	none; (7) imported (energy on R^3); (4), (5) imported into the box; cell energy bounded uniformly in t; energy inequality on a cell; pressure sublinear or bounded in x; pressure at most linear; zero mean velocity for all t	"In place of (4) and (5)"; the lists in (B), (D)
Mean of the (B) start	any; zero only	(8)
The erratum applied to (A), (C) too	box only; all alternatives	"made explicit in Eqn (8)"
Condition (4)	all derivatives; the field only	—
Condition (5)	joint decay in (x, t); separate	—

Clause	Readings	Departs from
Condition (7)	one bound for all t; finite at each t; energy inequality; with a pressure norm	—

Product of the numbers of readings, in the order of the table: $7 \cdot 4 \cdot 4 \cdot 4 \cdot 5 \cdot 2 \cdot 4 \cdot 4 \cdot 4 \cdot 3 \cdot 8 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 4 = 440{,}401{,}920$ combinations.

3. The reduction, and why it is valid

Order does not matter. Every reading on every clause restricts either the admissible data or the solution class, and a combination of readings is their conjunction. A conjunction does not depend on the order of its terms. So a verdict depends only on the set of readings chosen, not on whether the erratum's two sentences, or any other clauses, are applied in one order or another. Quantifier order, the one order that the wording fixes, is the viscosity clause.

Four clauses never change a verdict, and a fifth changes one group of readings. The clauses "where (1) holds", "force bounds", "condition (4)" and "condition (5)" never change a verdict (a factor of $2 \cdot 4 \cdot 2 \cdot 2 = 32$). The clause "condition (7)" changes a verdict only in the ordinary printed (C) and its variants, where three of its four readings give answered: yes (energy finite at each time only stays open); everywhere else it is irrelevant (a factor of 4). Together the five clauses contribute a factor of 128.

- Where (1) holds. For fields smooth up to $t = 0$, (1) at $t = 0$ follows from (1) for $t > 0$ by continuity. Under "smooth for $t > 0$ only" only $t > 0$ is meaningful, and weak solutions have no pointwise equation.
- Force bounds. Lean proves that natural and real exponents give the same force class (the equality of the two force classes). The Euclidean and componentwise norms differ by a factor of at most $\sqrt{3}$ (a Lean inequality; see the code block at the end of section 5). A bound with $C \geq 0$ gives one with $C > 0$.

- Conditions (4), (5), (7). These enter only (A) and (C). Condition (7) is the exception described above.

Monotonicity. If class $K \subseteq K\prime$ for the same data, then:

- a competitor in K is one in $K\prime$;
- "no competitor in $K\prime$" gives "no competitor in K";
- existence for every datum in K, as (B) asks, gives existence in $K\prime$.

Most of the reduction places each reading's class between two anchors:

- The slide-through escape (Lean, nanoda, lean4lean). For every admissible force, a printed-class competitor from rest exists. It is $C\hat{}\infty$ up to $t = 0$, its velocity is 1-periodic, its pressure is $\pi(x - Y, t) + \langle$the force's average $- Y'', x\rangle$ with π periodic, and its cell energy is bounded uniformly in t. So it lies in every class that drops pressure periodicity and adds nothing else (or adds only cell energy, linear pressure growth, or coarser periods).
- The forced breakdown with repeating pressure (Lean, nanoda, lean4lean). For every $\nu > 0$, OpenAI's force has no competitor that is periodic in velocity and pressure. So no subclass has one either.
- The chapter 6 class. Periodic pressure, velocity unrestricted: open for OpenAI's force, narrowed by the four literal-erratum lemmas (Lean; three of them also nanoda) and the two growth theorems of chapter 6 (Lean, nanoda). No competitor of additive-shear form exists (chapter 6; Lean, nanoda).

The remaining 3,440,640 combinations run through one decision list per column, below; the recipe for recounting is in the check guide. The earliest matching row decides. Each combination is decided exactly once, and each row is counted. The rows are the lemmas of section 5; the recount checks only the bookkeeping. Across the five columns there are 82 distinct patterns of verdicts (a pattern lists the verdict in each column; where a column's verdict depends on condition (7), as in the ordinary printed (C), that entry is the set of its verdicts). Rows that mix verdicts are split into sub-rows with their own counts, listed in the reading splits.

4. The reduced table

Counts are numbers of reading combinations. They give every reading equal weight, so they say nothing about which readings are reasonable. Rows that mix verdicts are split into sub-readings; each split, with the proof that decides it, is in the reading splits. A sub-reading whose verdict says "answered: yes in the trace weak class" or similar holds in the weak class that the closure proofs define (the class of weak solutions with a strong local L^2 initial trace; the closure proofs, section 1). Conditions use the clause names of section 2. "Periodic flow" means the erratum's label is not read as replacing (10). "Strong pressure" means the pressure must repeat for all t; or, with periodic flow, the extra condition is a sublinear pressure; or the test functions are one cell together with a binding identity (the pressure-periodicity lemma and the cell-test detection lemma). "Degenerate" means the identity-clause or the test-function clause turns an identity into one of $\partial_t u + \nabla p = 0$, $\partial_t u = 0$, $\nabla p = 0$ or curl $\partial_t u = 0$ (the extra-identity reduction).

(D), OpenAI's construction (their force, from rest), tested against every reading

Sonny (2026) gives each verdict in this table. The rows are in decision order: the earliest matching row decides.

Reading (earliest match decides)	**Verdict**	**Basis**	**Count**
(4), (5) imported as an extra condition	N/A	periodic data and force must vanish; OpenAI's force is outside the class	55,050,240
minimal period 1	N/A	rest has no minimal period	96,337,920
weak solutions, energy on R^3	Answered: yes	the whole-space energy exclusion and the curl of OpenAI's force	8,257,536
weak solutions, degenerate other than $\nabla p = 0$	Answered: yes	the frozen-vorticity obstruction (weak form) and the curl of OpenAI's force	24,772,608
weak solutions, $\nabla p = 0$	Open	not decided as a whole; global pressure freedom with periodic velocity is excluded in the Leray-Hopf refinement but not in the trace weak class; sub-cases in the reading splits and the closure proofs, section 6	9,289,728
weak solutions, zero mean velocity, periodic flow, strong pressure (no $\nabla p = 0$ requirement)	Answered: no	the construction's force has zero mean at every time (Lean), so a periodic Leray-Hopf solution from rest has zero mean velocity and is a competitor (cited existence; written, the weak zero-mean proof; the combining step is also checked in Lean, in two files named in the code block at the end of section 5: one takes both results as hypotheses, the other takes the force-mean result from its Lean proof and keeps the Leray-Hopf step as a named hypothesis)	645,120
weak solutions, otherwise	Answered: no	Leray (1934), Hopf (1951), cited: a global weak solution with periodic pressure	14,837,760
energy on R^3	Answered: yes	the whole-space energy exclusion and the curl of OpenAI's force	33,030,144
degenerate other than $\nabla p = 0$	Answered: yes	the extra-identity reduction, the frozen-vorticity obstruction and the curl of OpenAI's force; Lean, nanoda for 27,869,184 of these (the curl is Lean for every viscosity; solutions smooth up to $t = 0$, the text's tests and tests through time zero); written for the divergence-free force convention, divergence-free tests, one-cell tests and smoothness for $t > 0$ only	99,090,432
periodic flow, strong pressure, one force for all ν, no $\nabla p = 0$ requirement	Answered: no	a fixed periodic force has a global periodic-pressure solution from rest at large viscosity (written proof in the OpenAI-construction proofs)	6,881,280
periodic flow, strong pressure, one force for all ν, with $\nabla p = 0$	Open	a Lean equivalence (uniform-viscosity exclusion) reduces it exactly to excluding periodic forced Burgers competitors at every viscosity for one force; the construction supplies the matching viscosity only	9,289,728

Reading (earliest match decides)	**Verdict**	**Basis**	**Count**
periodic flow, strong pressure or $\nabla p = 0$, force with zero mean	Answered: yes	Lean: the construction's force has exactly zero cell mean at every time (the force-mean proofs)	8,085,504
periodic flow, strong pressure or $\nabla p = 0$	Answered: yes	the forced breakdown with repeating pressure; plus the positive-time exclusion (smooth for $t > 0$ only), the coarser-period reduction (period N), the pressure-periodicity lemma (sublinear pressure), the cell-test detection lemma (cell tests) and the divergence-free force reduction (divergence-free force) where the reading needs them; compact support in time uses OpenAI's Corollary 10.6 as stated	24,256,512
zero mean velocity, periodic flow, viscosity for each ν or one ν	Answered: yes	Lean (the zero-mean-velocity exclusion for periodic velocity) with the written energy and coarser-period proofs (the OpenAI-construction proofs); only the zero force mean is needed	3,440,640
zero mean velocity, one force for all ν, no $\nabla p = 0$ or cell-pressure requirement	Answered: no	the large-viscosity periodic-pressure solution is a competitor of this kind (written proof, same file)	2,580,480
zero mean velocity, otherwise (free velocity with one viscosity, or $\nabla p = 0$ readings with one force for all ν)	Open	see the zero-mean-velocity question in section 6; for free velocity the cell boundary flux is not controlled (the mean-balance argument is for periodic velocity only)	3,268,608
energy inequality, periodic flow, viscosity for each ν or one ν	Answered: yes	written energy and coarser-period proofs (the OpenAI-construction proofs): the energy identity forces mean velocity and tilt to vanish, then the forced breakdown with repeating pressure applies; a Lean energy-mean module checks the last algebra	3,440,640
energy inequality, one force for all ν, no $\nabla p = 0$ or cell-pressure requirement	Answered: no	the large-viscosity periodic-pressure solution is a competitor of this kind (written proof, same file)	2,580,480
energy inequality, otherwise (free velocity with one viscosity, or $\nabla p = 0$ readings with one force for all ν)	Open	see the cell-energy-inequality question in section 6; for free velocity the cell boundary flux is not controlled (the mean-balance argument is for periodic velocity only)	3,268,608
flow free, one-cell identity, viscosity for each ν or one	Answered: yes	written proof in the literal-erratum proofs (the one-cell boundary, derivative-matching and one-cell exclusion theorems): the	2,064,384

Reading (earliest match decides)	Verdict	Basis	Count
ν, no extra force convention, divergence-free projection or compact support		identity on arbitrary periodic vector tests on one cell forces matching traces and derivatives, so a periodic extension arises and the forced breakdown with repeating pressure excludes it; the Lean results on the period-defect system and face algebra check part of it	
the same, with the force required to have zero mean (viscosity for each ν or one ν)	Answered: yes	the same proof applies to the construction's own force, which has zero mean at every time (Lean); the one-cell-identity qualifications, part 1	688,128
the same, with one force for all ν and the corrected identity binding in full	Answered: no	a fixed smooth periodic force has a periodic-pressure solution at large viscosity (written proof in the OpenAI-construction proofs); it satisfies the identity on one cell (the one-cell-identity qualifications, part 2)	688,128
the same, with one force for all ν and only the time sign of the identity corrected ($\nabla p = 0$ on the cell)	Open	see the one-cell-identity question in section 6; excluding periodic forced Burgers flows at every viscosity for one force	688,128
flow free, $\nabla p = 0$	Open	subclass of the chapter 6 class	4,128,768
flow free, pressure always periodic	Open	the literal-erratum question of chapter 6; the literal-erratum proofs give controls and obstructions but neither a competitor nor an exclusion; every additive-shear competitor is excluded (Lean, nanoda)	2,359,296
flow free, extra condition: pressure sublinear	Open	see the sublinear-pressure question in section 6 (exact pressure criterion, written theorem in the literal-erratum proofs)	1,474,560
pressure periodic at $t = 0$ only	Answered: no	tilt removal while the force is zero (written; the zero window is Lean)	3,981,312
smoothness: smooth zero extension	Answered: no	tilt removal while the force is zero	3,981,312
otherwise (pressure free or modulo affine)	Answered: no	the slide-through escape	11,943,936

(D), the statement

Reading (earliest match decides)	**Verdict**	**Basis**	**Count**
(4), (5) imported as an extra condition	Answered: no	data and force are zero; $u = p = 0$ solves	55,050,240
zero mean velocity	Answered: yes	the witness forces for (D), zero-mean case (Lean, nanoda: starts of least period one, solutions smooth up to $t = 0$, smooth only for $t > 0$, the erratum's class and the flow-free class)	55,050,240
energy on R^3	Answered: yes	the whole-space energy exclusion (for (D): a nonzero repeating start has infinite energy; Lean, nanoda, including the weak readings)	55,050,240
weak, $\partial_t u = 0$	Answered: yes	the witness forces for (D), stationary case, weak form	10,321,920
weak, $\partial_t u + \nabla p = 0$ or curl $\partial_t u = 0$, period not minimal	Answered: yes	the frozen-vorticity obstruction, weak form	12,902,400
weak, zero-extended identity, minimal period	Answered: yes	a nonzero start cannot satisfy it	2,150,400
weak, remaining degenerate identities, with a strong local L^2 initial trace as in the trace weak class	Answered: yes	Lean, nanoda (trace weak class) for the identities on the text's test fields, the four weak degenerate identities stated directly and the identity tested through time zero; written for divergence-free-only or one-cell tests (the closure proofs, section 3): a primitive crossed-shear start with zero force violates the pressure-free or frozen-vorticity compatibility equation, for every viscosity	12,902,400
weak, otherwise	Answered: no	Leray (1934), Hopf (1951), cited	16,773,120
smoothness: smooth zero extension	Answered: yes	the witness forces for (D), zero-extension case (Lean, nanoda: starts of least period one, solutions smooth up to $t = 0$, smooth only for $t > 0$, the erratum's class and the flow-free class)	55,050,240
zero-extended identity, minimal period	Answered: yes	as the weak case with a zero-extended identity and minimal period (Lean, nanoda)	9,031,680
$\partial_t u = 0$ or $\nabla p = 0$	Answered: yes	the witness forces for (D), stationary and gradient cases; where a written witness is not admissible (the gradient force, a pure gradient, under the divergence-free force convention; either zero start when the start must have least period one; the shear force under compact support in time), the primitive start with zero force (the closure proofs, section 3), which breaks both the pressure-free and the frozen-vorticity compatibility equations at an interior point of the cell, for every ν and every force convention; Lean, nanoda for	56,770,560

Reading (earliest match decides)	Verdict	Basis	Count
		36,126,720 of these (with that start); written for the one-cell tests and the rest	
$\partial_t u + \nabla p = 0$ or curl $\partial_t u = 0$, period not minimal	Answered: yes	the frozen-vorticity obstruction; Lean, nanoda for 30,965,760 of these; written for the one-cell tests and the rest	38,707,200
minimal period 1 and a degenerate identity with a primitive start, or periodic flow with strong pressure and viscosity for each ν or one ν	Answered: yes	written (the closure proofs, sections 3 and 7) with the forced breakdown with repeating pressure; a single zero force works for every viscosity in the degenerate cases (degenerate part: Lean, nanoda for 7,741,440 of these)	13,148,160
minimal period 1, otherwise (a free pressure tilt, unrestricted velocity, or one force for all ν)	Open	exact controlled or unrestricted reductions in the closure proofs, section 7	8,785,920
OpenAI's construction meets the reading (the table above)	Answered: yes	OpenAI's construction, tested by Sonny (2026), with the split verdicts above	12,644,352
otherwise	Open	no witness known; for the printed wording this is the question whether (B) fails (Tao 2011, Proposition 1.7)	26,062,848

(B)

Reading (earliest match decides)	Verdict	Basis	Count
(4), (5) imported as an extra condition	Answered: yes	only $u^\circ = 0$ remains	55,050,240
weak, with energy on R^3, zero-extended identity, zero mean with any start, $\partial_t u = 0$, or ($\partial_t u + \nabla p = 0$ or curl $\partial_t u = 0$ with periodic flow)	Answered: no	the whole-space energy exclusion, the shear-start obstruction and the witness forces for (D) (zero-mean case), weak forms (Lean, nanoda for 42,061,824 of these in the trace weak class; written for the rest)	48,685,056
weak, remaining degenerate identities, with a strong local L^2 initial trace as in the trace weak class	Answered: no	Lean, nanoda (trace weak class) for the identities on the text's test fields, the four weak degenerate identities stated directly and the identity tested through time zero; written for divergence-free-only or one-cell tests (the closure proofs, section 3): the same primitive start has no competitor, without requiring velocity periodicity	11,354,112
weak, otherwise	Answered: yes	Leray (1934), Hopf (1951), cited; departs from "smooth functions"	17,031,168
energy on R^3	Answered: no	a nonzero periodic start has infinite energy (Lean, nanoda)	44,040,192
smoothness: smooth zero extension	Answered: no	forces $u(\cdot, 0) = 0$ (Lean, nanoda)	66,060,288
zero-extended identity	Answered: no	forces $u(\cdot, 0) = 0$ (Lean, nanoda)	43,352,064
zero mean velocity, mean of the start any	Answered: no	a start with nonzero mean fails at $t = 0$ (Lean, nanoda)	12,902,400
degenerate	Answered: no	the shear-start obstruction; Lean, nanoda for 56,770,560 of these (smooth up to $t = 0$ and smooth only for $t > 0$, the text's tests, divergence-free tests and tests through time zero); written for one-cell tests	90,832,896
periodic flow	Open	the classical (B) question; the readings agree by the Lean equivalence of printed and periodic-pressure existence and the equivalence lemma for (B)	39,739,392
flow free	Open	literal (B), the classical-questions item of section 6; velocity-only pressure-circulation system and one-cell reduction in the literal-erratum proofs (three of its theorems)	11,354,112

(A)

Reading (earliest match decides)	Verdict	Basis	Count
weak, erratum applied to all alternatives, with a pressure-free or frozen-vorticity identity, a binding zero-extended identity, or a periodic positive-time pressure Laplacian	Answered: no	Lean, nanoda (trace weak class) for the degenerate identities on the text's test fields, the identity tested through time zero, and a pressure that repeats at every positive time with the energy bound uniform in time (a cut-off primitive start in place of the swirl); written for the rest (the closure proofs, sections 2 and 4): a decaying swirl start has no competitor	35,684,352
weak, erratum applied to all alternatives, nondegenerate cell identity, or pressure only at time zero	Open	exact torus-extension reduction (answered: no in the Leray-Hopf refinement); an assigned pressure value at zero gives the opposite verdict to a genuine pressure trace (the closure proofs, section 5)	2,359,296
weak	Answered: yes	Leray (1934), cited; departs from "smooth functions" and (6)	50,036,736
smoothness: smooth zero extension	Answered: no	forces $u(\cdot, 0) = 0$ (Lean, nanoda)	88,080,384
erratum applied to all alternatives, degenerate or zero-extended identity	Answered: no	the shear-start obstruction (swirl start); Lean, nanoda for 70,189,056 of these; written for one-cell tests and the rest	94,961,664
erratum applied to all alternatives, pressure periodic (all t, modulo affine, or at $t = 0$ with smoothness at $t = 0$)	Answered: no	the pressure-on-all-of-space lemma: Lean, nanoda for 13,172,736 of these (smooth up to $t = 0$, and smooth only for $t > 0$ with the pressure repeating at every positive time); written for the other 2,162,688 (smooth only for $t > 0$, pressure repeating up to a constant or modulo affine functions; the closure proofs, section 4)	15,335,424
erratum applied to all alternatives, pressure at $t = 0$ only, smooth for $t > 0$ only	Answered: no	written (the closure proofs, section 8), when velocity and pressure are both continuous at time zero: a periodic initial pressure Laplacian contradicts the decaying swirl	1,769,472
erratum applied to the box only, with printed (12) also binding, or only the time sign or only the pressure sign of the identity corrected, imposed on (A) itself; the text's test fields; smooth up to $t = 0$, or the restriction of a C^∞ function on $R^3 \times R$	Answered: no	Lean, nanoda (Clay's 2000 wording stated directly; section 5): the extra-identity reduction turns the identity into $\partial_t u + \nabla p = 0$, $\nabla p = 0$ or $\partial_t u = 0$, and a compactly supported start equal to the shear (or the crossed shear) on the unit ball has no solution in Clay's (A) class; the restriction reading is a subclass. The clauses set aside in section 3 are handled as there (written)	16,515,072

Reading (earliest match decides)	Verdict	Basis	Count
otherwise	Open	the classical (A) question	135,659,520

The last row also holds readings that add a condition the text does not contain. Among them are the identity readings of (A) with the erratum in the box only that the Lean result does not cover: smoothness for $t > 0$ only, and tests that are divergence-free only, reach $t = 0$ with zero extension, or live on one cell. The covered ones are in the row above.

(C)

Reading (earliest match decides)	Verdict	Basis	Count
weak, erratum applied to all alternatives, with a pressure-free or frozen-vorticity identity, or a periodic positive-time pressure Laplacian	Answered: yes	Lean, nanoda (trace weak class) for the degenerate identities on the text's test fields, the identity tested through time zero, and a pressure that repeats at every positive time with the energy bound uniform in time (a cut-off primitive start in place of the swirl); written for the rest (the closure proofs, section 4; the weak-class (C) proofs, for OpenAI's actual force)	26,050,560
weak, erratum applied to all alternatives, a binding zero-extended identity	Answered: yes	Lean, nanoda; written proof (the zero-extended identity proof): a nonzero compactly supported start with zero force has no weak solution, in the trace weak class and in the usual weak class alike; the result that OpenAI's own force has a usual weak competitor concerns that force, not the statement	9,633,792
weak, erratum applied to all alternatives, a nondegenerate cell identity, or pressure only at time zero	Open	as for (A): see the closure proofs, section 5	2,359,296
weak	Answered: no	Leray (1934); Fefferman, p. 4, Theorem (B) (cited)	50,036,736
smoothness: smooth zero extension	Answered: yes	the witness forces for (D), zero-extension case, on R^3	88,080,384
erratum applied to all alternatives, degenerate	Answered: yes	the frozen-vorticity obstruction and the witness forces for (D), on R^3; Lean, nanoda for 66,060,288 of these; written for one-cell tests and the rest	90,832,896
erratum applied to all alternatives, pressure periodic as for (A)	Answered: yes	the pressure-on-all-of-space lemma: a decaying start with zero force has no competitor. Lean, nanoda for 14,745,600 of these (as for (A)); written for the other 2,457,600 (the closure proofs, section 4)	17,203,200
one common force for all ν (the start may or may not also be common)	Open as a group: answered: no if the start is also common (cited small-data theory), open if the start may depend on the viscosity	meaning 1 (one start and one force): Lean, assuming the cited small-data theorem (nanoda accepted an export of 53,132 declarations; built from the shipped build file); meaning 2: open, narrowed in Lean (nanoda: 33,099) and in writing (the common-force note); OpenAI's displayed scaled family is not one common force (the reading limits for (C), item 1) but that concerns the construction, not the statement (C)	52,068,352

Reading (earliest match decides)	Verdict	Basis	Count
ordinary printed (C), all clauses (1)-(7), force convention none or compact support, data chosen for each viscosity, one uniform whole-space energy bound	Answered: yes	Lean (the whole-space breakdown; nanoda accepted 94,908 declarations; the formal checks for (C)) and written extensions (the positive-time smoothness proof)	13,017,088
the same, condition (7) read as the usual energy inequality	Answered: yes	Lean, nanoda; written proof (the positive-time smoothness proof, section 5): for the construction's force, which has compact support in space and time, the inequality gives a uniform bound	13,017,088
the same, condition (7) read as the energy bound kept with a norm of the pressure added	Answered: yes	Lean, nanoda; written proof (the printed (C) variants): a subclass of the excluded class; assumes the energy bound is kept	13,017,088
the same, with the force required to have zero integral over all of space, condition (7) read in any of these three ways	Answered: yes	Lean, nanoda; written proof (the zero-integral force proof): a force with zero integral is chosen, and (C) asks only that some force exist	19,525,632
otherwise: energy finite at each time only, or a divergence-free force required	Open	the explicit limits are in the reading limits for (C) (second and third items) and the printed (C) variants, settlements 4 and 5	45,559,808

Each column sums to 440,401,920. In the ordinary printed (C) rows the four readings of condition (7) have equal weight. Three of them (one uniform bound, the usual energy inequality, the energy bound with a pressure norm added) give answered: yes for the force conventions none, compact support in time and zero integral; the reading with energy finite at each time only, and the divergence-free force convention, stay open. The one-common-force row (52,068,352 readings) is counted open as a group: one meaning (one start and one force) is answered: no, and the other (one force, the start chosen for each viscosity) is open (the common-force note).

5. Lemmas

Labels: Lean (with the checkers named); written (complete argument below; the limits of review are in the check guide); cited (a published theorem, not rechecked here).

The smooth-solutions identity (written; Lean, nanoda, for solutions smooth up to $t = 0$ with any smooth force). If (u, p) is smooth for $t > 0$ and satisfies (1), (2) there, the erratum's identity holds for every test of the text. Proof. Multiply (1) by θ, integrate over $R^3 \times (0, \infty)$, and move the derivatives onto θ. There are no boundary terms, because θ is compactly supported in that open set. The convection term uses (2). ■ Requiring the identity in place of (10), or beside it, therefore adds nothing for smooth solutions. Chapter 6 also checked the signs numerically.

The extra-identity reduction (written; the quantified test identities (a) to (c) are Lean; for smooth solutions and the text's test fields the reduction itself is Lean). Write $A = \iint u \cdot \partial_t \theta$, $B = \Sigma \iint u_i u_j \partial_j \theta_i$, $L = \nu \iint u \cdot \Delta \theta$, $F = \iint f \cdot \theta$ and $P = \iint p \operatorname{div} \theta$. By the smooth-solutions identity, $-A - B = L + F + P$. So, with the text's tests:

- (a) Printed (12), $A - B = L + F - P$, holds for all θ iff $A + P = -\iint (\partial_t u + \nabla p) \cdot \theta = 0$ for all θ, that is, iff $\partial_t u + \nabla p = 0$.
- (b) With only the time sign corrected, $-A - B = L + F - P$ iff $P = 0$, that is, iff $\nabla p = 0$.
- (c) With only the pressure sign corrected, $A - B = L + F + P$ iff $A = 0$, that is, iff $\partial_t u = 0$.

Other test conventions:

- Divergence-free tests only. P vanishes, so (a) and (c) become curl $\partial_t u = 0$ and (b) holds for every solution.
- Tests reaching $t = 0$, with zero extension. Each identity gains the term $\int u(x, 0) \cdot \theta(x, 0)\, dx$. The erratum's identity then also forces $u(\cdot, 0) = 0$. Tests supported in $t > 0$ still give (a)-(c).
- Periodic tests on one cell. See the cell-test detection lemma for periodic u. For non-periodic u, tests supported inside the open cell give (a)-(c) on that cell only.

The curl of OpenAI's force (written; Lean, nanoda, for every viscosity: the periodic-potential premise of the conditional Lean form is proved for forces smooth on all of space-time, which OpenAI's force is). OpenAI's force f satisfies curl $f \not\equiv 0$. Proof. Suppose curl $f \equiv 0$. Let $m(t)$ be the cell mean of f and put $g = f - m$. Define $\varphi(x, t)$ as the line integral of g from 0 to x.

- φ is well defined, because g is curl-free on R^3.
- φ is smooth in (x, t) up to $t = 0$.
- The x-gradient of $\varphi(x + e_j, t) - \varphi(x, t)$ is $g(x + e_j) - g(x) = 0$. So the difference is constant in x, and integrating over the cell shows the constant is the cell mean of g_j, which is 0. Hence φ is periodic.

Then $u(x, t) = \int_0^t m$, $p = \varphi$ is smooth, periodic in velocity and pressure, starts from rest, and solves (1), (2): $\partial_t u = m$ and $\nabla p = f - m$. That contradicts the forced breakdown with repeating pressure. ■

The frozen-vorticity obstruction (written; the smooth case is Lean). Let f be smooth with curl $f \not\equiv 0$, and let (u, p) start from rest. Any one of the following conditions rules out a solution, with any velocity or pressure condition:

- $\partial_t u = 0$. Then $u \equiv 0$, so $\nabla p = f$, which is curl-free.
- $\partial_t u + \nabla p = 0$ or curl $\partial_t u = 0$.

On one cell, the same holds on that cell, and periodicity of f carries curl $f = 0$ everywhere. Weak form: $\partial_t \omega = 0$ in distributions, and weak continuity at $t = 0$ gives $\omega \equiv 0$. Then u is harmonic in x (Weyl's lemma) and the same identities follow. ■ With the curl of OpenAI's force, this decides every degenerate reading for OpenAI's force, except $\nabla p = 0$ with velocity periodicity dropped (open; see the reduced table).

The shear-start obstruction (written; the four periodic obstructions are Lean; the whole-space form is Lean with a cut-off shear start; values checked numerically). Explicit starts with no solution:

- $\partial_t u = 0$. The shear $u^\circ = (\sin 2\pi x_2, 0, 0)$ gives $u \equiv u^\circ$. Then $(u^\circ \cdot \nabla) u^\circ - \nu \Delta u^\circ = 4\pi^2 \nu\, u^\circ$ would have to be a gradient, but its curl is nonzero.

- $\partial_t u + \nabla p = 0$. For the same shear, subtracting this condition from (1) leaves $(u\cdot\nabla)u - \nu\Delta u = 0$. At $t = 0$ this is $4\pi^2\nu\, u^\circ \neq 0$.
- curl $\partial_t u = 0$. The curl of (1) at $t = 0$ gives $\nu\Delta\omega^\circ = \mathrm{curl}((u^\circ\cdot\nabla)u^\circ) = 0$, but $\Delta\omega^\circ = -4\pi^2\omega^\circ \neq 0$.
- $\nabla p = 0$. Take the divergence of (1) at $t = 0$. For $u^\circ = (\sin 2\pi x_2, \sin 2\pi x_1, 0)$ the term $\partial_i\partial_j(u^\circ_i u^\circ_j) = 8\pi^2 \cos 2\pi x_1 \cos 2\pi x_2$ would have to vanish.
- On $\mathbb{R}^3$, the swirl $u^\circ = (-x_2, x_1, 0)e^{-|x|^2}$ does the same in each case: each quantity is nonzero at sample points.
- Smooth for $t > 0$ only. The same conclusions hold, because u is continuous at $t = 0$ and the quantities above converge in distributions.

- Weak forms. For $\partial_t u = 0$, $u \equiv u^\circ$ is smooth. For $\partial_t u + \nabla p = 0$ or curl $\partial_t u = 0$ with periodic u, the difference $u - u^\circ$ is a periodic harmonic gradient, hence a constant $c(t)$, and $((u^\circ + c)\cdot\nabla)u^\circ = \nu\Delta u^\circ$ fails for the shear. ■

The witness forces for (D) (written; the smooth-zero-extension case, the ordinary case of the zero-mean case and the force admissibility of the stationary case are Lean; the zero-mean case with starts of least period one, and with a zero-mean start, is Lean, nanoda). Witnesses for the statement (D):

- $\partial_t u = 0$ or $\nabla p = 0$.
- Smooth zero extension. All derivatives vanish at $t = 0$, so (1) forces $f(\cdot, 0) = 0$. Any force with $f(\cdot, 0) \neq 0$ has no solution.

- Zero mean velocity for all t. The start $u^\circ = e_1$ is smooth, divergence-free and periodic, and violates the condition at $t = 0$. Under "minimal period 1", use $(1 + \sin 2\pi x_2 + \sin 2\pi x_3, \sin 2\pi x_1, 0)$.
- These forces satisfy (8), (9), because every derivative decays like e^{-t}. ■

The whole-space energy exclusion (written; the growth contradiction is Lean, and the whole is Lean conditional on its cutoff estimates). Let f be smooth

and periodic with curl f $\not\equiv 0$. Then no (u, p), smooth or weak, has sup_t $\int_{R^3}|u|^2 < \infty$ and satisfies the erratum's identity.

- Choice of test. Pick $t_1 > 0$ with curl $f(\cdot, t_1) \not\equiv 0$. Put $g = \text{curl } f(\cdot, t_1)$, and choose $\psi \geq 0$ concentrated near t_1 so that $\int\psi\langle \text{curl } f, g\rangle_{\text{cell}}\, dt \neq 0$. Take $\theta = \psi(t)\,\text{curl}(\varphi(x/R)\, g(x))$. It is divergence-free, so the pressure term vanishes for any pressure.
- Left side. The velocity terms are $O(R^{3/2})$ by Cauchy-Schwarz with the energy bound, and the convection term is $O(1)$.
- Force term. It equals $\iint\psi\varphi(x/R)\langle \text{curl } f, g\rangle = cR^3 + O(R^2)$ with $c \neq 0$.
- Conclusion. Large R gives a contradiction. ■

With the curl of OpenAI's force this applies to OpenAI's force. In the periodic class, finite energy on R^3 forces $u \equiv 0$, so a nonzero (B) start is excluded.

Tilt removal while the force is zero (written; OpenAI's zero window, gauge removal and zero extension are Lean; the whole is Lean conditional on the translated-zero-window premise, which is steps 1 to 3 below). OpenAI's force vanishes for $0 \leq t \leq t_0$ for some $t_0 > 0$ (Lean: the construction keeps a positive initial interval on which its force is zero; the time-slab item of chapter 6). Then OpenAI's force has a printed-class competitor (u, p) from rest with $u \equiv 0$ and $p \equiv 0$ for $t \in [0, \varepsilon]$. In particular its pressure is periodic at $t = 0$, and its zero extension is C^∞.

- Step 1: remove the tilt near $t = 0$. Let (u, p) be the competitor of the slide-through escape, with tilt $a(t) =$ the force's average $- Y''$. Choose a smooth Z with $Z = \chi(t)\int_0^t\int_0^s a$, where $\chi \equiv 1$ on $[0, \varepsilon]$ and $\chi \equiv 0$ for $t \geq t_0/2$. Put $u_Z(x, t) = u(x - Z, t) + Z'(t)$ and $p_Z = p(x - Z, t) - Z''(t)\cdot x$.
- Step 2: it is still a competitor. The pair (u_Z, p_Z) solves (1), (2) with force $f(x - Z(t), t)$. That equals f, because $Z \neq 0$ only where $f = 0$. It starts from rest, its velocity is periodic, and its tilt is zero on $[0, \varepsilon]$.
- Step 3: it vanishes near $t = 0$. On $[0, \varepsilon]$ the pair is periodic in velocity and pressure with zero force, so $d/dt\, \|u_Z\|^2_{\text{cell}} = -2\nu\|\nabla u_Z\|^2 \leq 0$

and $u_Z \equiv 0$. Then $\nabla p_Z = 0$. Subtracting the function $p_Z(0, t)$, which is a gauge, makes $p_Z \equiv 0$ on $[0, \varepsilon]$. ■

The positive-time exclusion (written; the general positive-time exclusion is Lean; inputs: OpenAI's Lean candidate properties and Corollary 10.6). The exclusion of the forced breakdown with repeating pressure also holds when the competitor is C^∞ only for $t > 0$ and continuous on $\mathbb{R}^3 \times [0, \infty)$. Proof.

- Let U be OpenAI's solution. It is smooth on $[0, 1)$ and periodic, and $\|U(t)\|_\infty$ is unbounded as $t \uparrow 1$. Put $w = u - U$.
- Periodic integration removes the transport and pressure terms, so $\frac{d}{dt} \|w\|^2 \le 2 \sup_{[0,T]} \|\nabla U\|_\infty \|w\|^2$ on $(0, T]$ for every $T < 1$.
- Uniform continuity on the compact cell gives $\|w(s)\| \to 0$ as $s \to 0$, so $w \equiv 0$ on $[0, 1)$.
- But u is continuous on $[0, 1]^3 \times [0, 1]$, so it is bounded there, which contradicts the blowup of U. ■

The coarser-period reduction (written, partly Lean). Let N be a positive integer, and keep the same 1-periodic force and the same periodic start. If (u, p) is a competitor whose velocity and pressure have period N (lattice $N\mathbb{Z}^3$), then it is 1-periodic.

- $(u(\cdot + e_j), p(\cdot + e_j))$ is also a competitor, since f is 1-periodic. The energy argument on the $N\mathbb{Z}^3$ torus makes the two equal.
- So u is 1-periodic, and $p(x + e_j) - p(x) = c_j(t)$. N-periodicity of p gives $N c_j = 0$. ■
- The forced breakdown with repeating pressure then applies. With periodic velocity, the same shift argument shows that printed (B) with period-N solutions is the same as with period 1.

The pressure-periodicity lemma (Lean).

- (a) If u is periodic, then $\nabla p = f - \partial_t u - (u \cdot \nabla) u + \nu \Delta u$ is periodic. So "pressure modulo affine functions" adds nothing to the printed class.
- (b) In that case $p(x + e_j, t) - p(x, t) = c_j(t)$. If p is sublinear in x at each t, then $c_j = 0$, so p is periodic. ■

The cell-test detection lemma (written; the finite detection step is Lean). For periodic u and spatially periodic tests on the unit cell, the boundary terms of the pressure integral leave the tilt term $\tau(\theta) = \Sigma_j\, a_j \int_{face_j} \theta_j$. Each identity then holds iff its condition from the extra-identity reduction holds and $a \equiv 0$. Tests supported inside the cell give that condition, and then $\tau = 0$ for all θ gives $a = 0$. So the pressure is periodic. ■

The divergence-free force reduction (proved in writing (a second reading within this project), in the divergence-free force proof; the residual, printed-solution and pressure-periodicity equivalences are Lean). On the torus only. Let f be smooth up to $t = 0$ and satisfy (8), (9). Let $\varphi(\cdot, t)$ be the mean-zero periodic solution of $\Delta\varphi = \operatorname{div} f(\cdot, t)$, and put $f' = f - \nabla\varphi$. Then:

- f' is smooth, periodic, divergence-free and satisfies (8), (9) (Fourier bounds). It has the same curl and the same cell mean as f. It is zero at every time at which f is identically zero, so a zero window or compact support in time is kept. On all of space the projection breaks (5) (the printed (C) variants, settlement 5), the lemma applies on the torus.
- (u, p) solves (1), (2) with f iff $(u, p - \varphi)$ solves them with f'. This map keeps every velocity condition and every smoothness reading. The smooth zero extension is kept when f vanishes to all orders at $t = 0$, as OpenAI's force does. The map also keeps every pressure reading except $\nabla p = 0$: repeating at all times, up to a function of t, modulo affine functions, at $t = 0$ only, sublinear, at most linear, bounded. $\nabla p = 0$ carries over one way only: if $\nabla(p - \varphi) = 0$, then p repeats at all times.

- Energy, and the boundary condition. For $\operatorname{div} u = 0$ the gradient work on a cell is $\int_Q \nabla\varphi \cdot u = \oint_{\partial Q} \varphi\, u \cdot n$. It vanishes when the velocity is periodic. So the cell energy equality and inequality carry over for periodic velocity. Nothing is claimed for free velocity.
- Identities. Two identities add the pressure term: the erratum's, and the one with only the pressure sign corrected. They carry over for the text's tests, for tests through $t = 0$ with zero extension, and for periodic tests on one cell, with any velocity. Two subtract it: printed (12), and the one with only the time sign corrected. They change by $-2\iint \nabla\varphi \cdot \theta$ and do

not carry over, except on divergence-free tests, where every identity does. Under those readings f′ is decided directly, since the extra-identity reduction holds for every smooth force. ■

So under the divergence-free-force convention, OpenAI's column is read for f′. Each of its verdicts holds for f′ by one of three routes:

- the row's own argument applied to f′;
- the map back to f, for exclusions;
- the map to f′, for competitors.

The divergence-free force proof, section 5, lists the route for each row. No row needs a transfer that the lemma excludes.

The equivalence lemma for (B) (written; with the Lean equivalence of printed and periodic-pressure existence, Lean; the printed-to-periodic-pressure step is also Lean). For (B) with periodic flow:

- Printed versus erratum. Printed existence for u° is equivalent to existence with periodic pressure (the Lean equivalence, built in the book's default target; checked by nanoda and lean4lean). So pressure-at-$t = 0$-only, modulo-affine, sublinear and linear pressure readings give the same question.
- Mean of the start. A start with mean c reduces to mean 0 by the Galilean change $v(x, t) = u(x + ct, t) - c$, which keeps both periods.
- Energy readings. The periodic-pressure solution satisfies the energy equality, so the cell-energy and energy-inequality readings give the same question. ■

The pressure-on-all-of-space lemma (Lean, nanoda, for solutions smooth up to $t = 0$ and for solutions smooth for $t > 0$ with the pressure repeating at every positive time; written for the rest). The Lean proof takes a shorter route than the written one and needs no decay step: the divergence of (1) at $t = 0$ gives $\Delta p(\cdot, 0) = -\operatorname{div}\operatorname{div}(u^\circ u^\circ)$; every pressure reading makes the left side repeat, while the right side vanishes outside a ball, so both vanish everywhere; but for the crossed-shear start on the unit ball the right side is

nonzero at the origin. Apply the pressure sentence to all of space. If $p(\cdot, 0)$ is periodic (or $\nabla p(\cdot, 0)$ is) and u° satisfies (4), then:

- $-\Delta p(\cdot, 0) = \partial_i \partial_j (u^\circ_i u^\circ_j) - \operatorname{div} f(\cdot, 0)$. The left side is periodic, even if only $\nabla p(\cdot, 0)$ is. The right side tends to zero at infinity, by (4) and (5): pointwise when (4) bounds every derivative, and in distributions (paired with a test translated far away, as in the closure proofs, section 4) when (4) bounds the field only.
- A periodic function or distribution that tends to zero at infinity in this sense is zero: translate any point, or the test, by $n\, e_1$ and let n tend to infinity.
- So $\partial_i \partial_j (u^\circ_i u^\circ_j) = \operatorname{div} f(\cdot, 0)$. The swirl start violates this with $f = 0$ (its left side is -2 at the origin), and any force with $\operatorname{div} f(\cdot, 0) \neq 0$ violates it from rest.

The complete written proof is in the closure proofs, section 4. It uses no Newtonian-potential or harmonic-remainder estimate. The same section treats a pressure that repeats at positive times only, using finite energy on all of space at each time, which covers the case "smooth for $t > 0$ only".

Cited.

- R. T. Seeley, Extension of C^∞ functions defined in a half space, Proc. AMS 15 (1964): "C^∞ up to $t = 0$" equals "restriction of a C^∞ function". Both readings of smoothness are therefore one class, Lean's notion of smooth on the closed half-space.
- J. Leray (1934) and E. Hopf (1951): global weak solutions. Fefferman's own page 4 states the whole-space case.
- T. Tao (2011), Proposition 1.7: the printed forced and unforced periodic questions are equivalent. That is the basis of the remaining readings of (D) for the printed wording, and it is not formally checked here.

Lean status of the written lemmas

Two Lean passes (62 and 138 theorem declarations; Lean 4.34.1, Mathlib d13f23b7; every printed axiom list is only propext, Classical.choice and

Quot.sound; nanoda accepted exports of 44,913 and 46,911 declarations containing every new theorem) formalised parts of the lemmas above. The sources are in the two Lean collections named in the exact statements page. A part proved in Lean is not counted as a proof of the whole lemma. Where a Lean result is conditional, the premise is a named proposition in the same file and is not proved. These two passes decide no row on their own: each row's verdict rests on the written argument or on the slide-through escape and the forced breakdown with repeating pressure. A third pass in the second round (six files in a separate build target; every printed axiom list is only the same three axioms; nanoda accepted an export of 93,579 declarations containing every new theorem) states Clay's (A), (B) and (D) under the 2000 wording directly against the comparator definitions, so the rows marked below as Lean in the second round are checked as statements about Clay's wording itself. A fourth pass in the second round (nine files in a further build target; the same three axioms; nanoda accepted an export of 94,345 declarations containing every new theorem) proves the smooth-solutions identity, the curl of OpenAI's force without its premise, the start-level readings with their witnesses and the pressure-on-all-of-space lemma by a shorter route; the rows it reaches say "Lean, nanoda, second round" with the number of readings reached.

Lemma	Lean status
The smooth-solutions identity	Lean, nanoda: for solutions smooth up to $t = 0$ with any smooth force, with the integrability conditions of the weak class; such solutions, and those smooth only for $t > 0$, lie in the trace weak class
The extra-identity reduction	the quantified test identities (a) to (c): Lean. For smooth solutions and the text's test fields, the reduction itself (the four integrations by parts, the fundamental lemma and the step to $t = 0$), for any smooth force: Lean. Divergence-free-test, zero-extension, initial-trace and one-cell forms: written
The curl of OpenAI's force	Lean, nanoda, for every viscosity, with no premise left; the uniform-mean solution is Lean
The frozen-vorticity obstruction	smooth case Lean: all three degeneracies, from rest, with no periodicity assumption. Weak form in the trace weak class: Lean; weaker traces: written
The shear-start obstruction	four periodic obstructions Lean (stationary shear; time-plus-pressure shear; curl-time shear; crossed shear with zero pressure gradient, every viscosity). Whole-space form: Lean, with a compactly supported start equal to the shear on the unit ball in place of the swirl (the claim is the same: some start has no solution). Positive-time-only form and divergence-free-test form: Lean, nanoda. Weak forms: Lean in the trace weak class; weaker traces: written
The witness forces for (D), stationary case	admissibility (8), (9) of both written exponential forces Lean; stationary-shear and gradient exclusions Lean. The exponential shear force is not compactly supported in time (Lean), so the compact-time readings use the primitive start with zero force (closure proofs, section 3; Lean), which is compactly supported in time
The witness forces for (D), zero-extension case	Lean
The witness forces for (D), zero-mean case	ordinary constant start Lean; the minimal-period witness and the zero-mean-start witness: Lean, nanoda, for four solution notions that take the start pointwise
The whole-space energy exclusion	growth contradiction Lean; the whole is Lean conditional on its cutoff estimates
Tilt removal while the force is zero	gauge removal and smooth zero extension Lean; OpenAI's zero window Lean; the whole is Lean conditional on a translated-zero-window premise (steps 1 to 3 of the written proof)
The positive-time exclusion	the general positive-time exclusion is Lean; the binding to OpenAI's exact force is written
The coarser-period reduction	finite period kills the jump and pressure has unit period: Lean. Shift uniqueness on the N torus: written
The pressure-periodicity lemma	Lean

Lemma	Lean status
The cell-test detection lemma	the finite linear detection step is Lean; face integrals and test duality: written
The divergence-free force reduction	proved in writing (a second reading within this project), the divergence-free force proof; residual, printed-solution and pressure-periodicity equivalences Lean; torus Poisson existence and bounds, and the pressure, energy and identity transfers: written
The equivalence lemma for (B)	partly Lean, using the equivalence of printed and periodic-pressure existence
The pressure-on-all-of-space lemma	Lean, nanoda for solutions smooth up to $t = 0$ (every pressure reading) and for solutions smooth for $t > 0$ with the pressure repeating at every positive time, by the route above; written for smoothness for $t > 0$ only with the pressure repeating up to a constant or modulo affine functions (the closure proofs, section 4)
The literal-erratum lemmas of chapter 6 (period-defect system, nonzero-defect lemma)	Lean (in the literal-erratum Lean folder). The bridge to the forced breakdown with repeating pressure has been checked by Lean and nanoda; it was rebuilt from source (standard axioms only; the replay tool leanchecker passed)
Clay's 2000 wording, stated directly	(B) answered: no and (D) answered: yes under energy on R^3, zero mean velocity, smooth zero extension, each pointwise degenerate identity, and each of the three binding identities with the text's test fields; (A) answered: no under smooth zero extension, the pointwise degenerate identities and the three binding identities, with the identity imposed on (A) itself: Lean. The (A) decision list of section 4 counts the identity readings of (A) with the erratum in the box only as answered: no where this result covers them (the text's test fields, smooth up to $t = 0$; 16,515,072 readings); the other test conventions and smoothness for $t > 0$ only stay in the open row. Weak-solution forms in the trace weak class: Lean. The other test conventions: written
Seeley, Leray-Hopf, Tao Proposition 1.7, where (1) holds (equations for $t > 0$)	not formalised (cited or written)

The Lean declarations behind these statuses are listed here, grouped by lemma. The Lean names are internal labels, not terms of the problem.

```
-- extra-identity reduction
--   NSContinuation.Wprime_printed_test_iff, Wprime_time_corrected_test_iff
--   module NSLemmas.IdentityAlgebra
-- curl of OpenAI's force (no premise left; files lean/WRCurl.lean, WRUnifo
--   WrittenRows.openai_force_curl_nonzero, potential_of_curl_free, openai_
--   NSContinuation.uniformPrimitive_solution   (the uniform-mean solution)
--   NSContinuation.H_conditional_on_periodic_potential   (the conditional
-- frozen-vorticity obstruction
--   NSContinuation.D1_smooth
-- shear-start obstruction
--   D2_shear_stationary, D2_shear_cancel, D2_shear_curl_time, D2_crossed_z
-- witness forces for (D)
--   forceProfile_admissible, D3a_shear_stationary_exclusion, D3a_gradient_
--   exponential_witness_not_compact_time
--   NSLemmas.D3b_zero_extension_force_zero
--   NSContinuation.D3b_admissible_witness_exclusion
--   NSLemmas.D3c_nonzero_mean_start, D3c_e1_start
-- whole-space energy exclusion
--   NSContinuation.E_growth_contradiction, E_conditional_on_cutoff_estimat
-- tilt removal
--   gauge_preserves_printed, zeroPast_smooth_of_zero_window, G_conditional
--   InitialGap.periodic_zero_mean_force_with_gap   (the construction's zer
-- positive-time exclusion
--   NSLemmas.PositiveTime.S_exclusion
-- coarser-period reduction
--   NSLemmas.N_finite_period_kills_jump, N_pressure_unit_period
-- pressure-periodicity lemma
--   NSLemmas.Q_gradient_periodic, Q_pressure_periodic, Q_printed_sublinear
-- cell-test detection lemma
--   NSLemmas.C4_detect_all_tilts
-- divergence-free force reduction
--   NSLemmas.Helm_residual_invariant, Helm_printed_solution_iff, Helm_pres
-- equivalence lemma for (B)
--   NSLemmas.M_printed_periodic_pressure, M_intermediate_existence
--   JustExploringInherited.inherited_existence_iff  (file InheritedBridge.
-- pressure-on-all-of-space lemma
--   NSLemmas.AP_periodic_poisson_source_zero, periodic_decays_zero
--   second-round route (file lean/WRPressure.lean; positive-time share in
--   WrittenRows.no_solution_cross_lap_periodic, A_pressure_all_times_not_n
--     A_pressure_gradient_periodic_not_met, A_pressure_at_zero_not_met, A_
-- smooth-solutions identity (solutions smooth up to t = 0; file lean/WRBri
--   WrittenRows.smooth_solutions_identity, traceWeak_of_smooth
--   WrittenRows.traceWeak_of_positive   (solutions smooth only for t > 0;
-- start-level readings and their weak forms (files lean/WRStart.lean, WRWe
--   WrittenRows.B_min_readings_not_met, D_min_readings_met, B_zero_start_r
```

```
--     trace_no_energy, trace_no_zero_mean
-- weak forms in the trace weak class (files lean/WC*.lean, build target Re
--   WeakClass.B_weak_degenerate_not_met_min, D_weak_degenerate_met_min, B_
--     D_weak_zero_extended_met, A_weak_degenerate_not_met, C_weak_degenera
--     A_weak_zero_extended_not_met, C_weak_zero_extended_met,
--     A_weak_periodic_pressure_not_met, C_weak_periodic_pressure_met
-- literal-erratum lemmas of chapter 6 (folder lean/NSClose/)
--   NSClose.defect_system, NSClose.competitor_carries_nonzero_defect
--   NSCloseBridge.openai_force_literal_competitors
-- weak zero-mean combination (file lean/Combine.lean; logic step only, the
--   NSRemaining.combine
--   CClosures.openai_weak_zero_mean_of_LH   (file lean/OpenAIWeakZeroMean.
--     the Leray-Hopf step is the named hypothesis)
-- Clay's 2000 wording, stated directly (files lean/OWDefs.lean, OWA.lean,
--   OWIdent.lean, OWSuper.lean; build target ReadingsOriginalWording)
--   solution class and bridges: OriginalWording.ClaySolution, claySolution
--     clayB_rest_iff, smoothSolution_of_clay; readings: ClayAWith, ClayBWi
--   (B): B_whole_space_energy_not_met, B_zero_mean_velocity_not_met, B_smo
--     B_stationary_not_met, B_pressure_cancels_not_met, B_curl_time_not_me
--     B_printed_identity_not_met_lean, B_time_corrected_identity_not_met_l
--     B_pressure_corrected_identity_not_met_lean
--   (A): A_smooth_zero_extension_not_met, A_stationary_not_met, A_pressure
--     A_curl_time_not_met, A_pressure_free_not_met, A_printed_identity_not
--     A_time_corrected_identity_not_met_lean, A_pressure_corrected_identit
--   (D): D_whole_space_energy_met, D_zero_mean_velocity_met, D_smooth_zero
--     D_stationary_met, D_pressure_cancels_met, D_curl_time_met, D_pressur
--     D_printed_identity_met, D_time_corrected_identity_met, D_pressure_co
--   extra-identity reduction: reduction_printed_force, reduction_time_corr
--     reduction_pressure_corrected_force, fourIntegrals, eq_zero_of_integr
-- other names used in section 3
--   ForcedBreakdown.printed_force_iff_admissible  (file ForcePrinted.lean,
--   euclidean_norm_le_sqrt_card  (file JustCheckin/Geometry.lean)
```

The exact form of seven lemma statements

The lemma texts above use these exact forms.

- The start-level witnesses. For the start-level readings the Lean witnesses are the primitive start plus a constant multiple of e_1, and the primitive start minus its cell mean, for four solution notions. They are admissible when the period must be exactly 1, they apply to the class whose velocity repeats, and they cover the reading "the (B) start has

zero mean". A constant start, or a zero start with a nonzero force, would not cover these readings.

- The coarser-period reduction. The lemma takes a positive integer N, the same 1-periodic force and the same periodic initial data. "Period N with N unspecified" is not enough.

- The divergence-free force reduction. The lemma is stated on the torus, with the boundary conditions that remove the gradient work. The gradient work on a cell is the face flux $\oint \varphi\, u{\cdot}n$, which vanishes for periodic velocity. Any binding identity is checked separately before a verdict is transferred: $\nabla p = 0$ carries over one way only, and printed (12) and the identity with only the time sign corrected do not carry over, so they are decided for f' directly (the divergence-free force proof, section 5).
- The pressure-on-all-of-space lemma. The route is: the periodic Laplacian is a decaying Poisson source, so it vanishes. The Poisson identity and the decay are hypotheses, not derived from the data assumptions.
- The witness forces for (D), compact time support. The exponential shear witness is not compactly supported in time (Lean). Under the reading "compact support in time" the witness is the primitive start with zero force (closure proofs, section 3; Lean), which is compactly supported in time.

- The zero-mean-velocity reading of OpenAI's construction. The mean-balance equivalence in the zero-mean-velocity question holds for the periodic-velocity branch only. The reading also has an unrestricted-velocity part, for which the cell boundary flux need not vanish; section 4 splits it into three parts.
- Heat shear. The heat shear is nontrivial at every positive time, because its value at $y = 0$ is positive. It is not nonzero at every point: exact rational enclosures at time $1/10$ give opposite signs at $y = 15/4$ and $y = 4$, so it has a zero between them. This affects neither the literal-erratum question, nor the sublinear-pressure question, nor the literal (B) reading (the literal-erratum status note).

Written results behind section 4

Each is a complete written argument within its stated assumptions, with the exact class stated in the file linked. Cited inputs (weak existence of Leray and Hopf, local theory) keep their labels. None changes the classical (A) question, the classical (B) question or the remaining readings of (D) as questions, although the remaining readings of (D) lose the readings that the OpenAI column now decides (section 4).

Result	Readings it settles	File
A primitive crossed-shear start with zero force violates the compatibility equation of the pressure-free or frozen-vorticity identity; the frozen-vorticity branch needs only a weak local L^2 initial trace	(D) with weak solutions and a remaining degenerate identity: answered: yes. (B), the same readings: answered: no. (D) with a start of minimal period one, degenerate part: answered: yes	the closure proofs, sections 3 and 7
A decaying swirl start has no competitor under a periodic positive-time pressure Laplacian, a degenerate identity or a binding zero-extended identity	(A) with weak solutions and the erratum applied to all of space, part: answered: no. (C), the same, part: answered: yes	the closure proofs, section 4
Continuous velocity and pressure at time zero with periodic initial pressure contradict the decaying swirl	(A) with the pressure repeating at time zero only and smoothness only afterwards: answered: no	the closure proofs, section 8
Weak pressure at zero: an assigned value and a genuine trace give opposite verdicts; the one-cell identity reduces exactly to a torus extension	(A) and (C) with weak solutions and the erratum applied to all of space (sub-readings left open)	the closure proofs, section 5
OpenAI's construction with weak solutions and pressure freedom: a constrained Burgers comparison in the Leray-Hopf class; a force-mean criterion for the zero-mean subclass	Open as a whole	the closure proofs, section 6
(D) with a start of minimal period one: a preparation step with the primitive start and the forced breakdown with repeating pressure	Answered: yes for the part so described	the closure proofs, section 7
The construction's force has exactly zero mean at every time (Lean)	the zero-mean-force reading: answered: yes	the force-mean proofs
Zero-mean and cell-energy-inequality competitors are excluded for periodic velocity at the matching viscosity; a fixed force has a periodic-pressure solution at large viscosity; a pressure-free reduction to forced Burgers (Lean equivalence); a zero-force, zero-cell-mean pressure-free flow that is not periodic	one force for every viscosity, zero mean velocity and cell energy inequality: split as in section 4; the pressure-free readings reduced	the OpenAI-construction proofs
One-cell identity on arbitrary vector tests forces matching traces and derivatives, so the fixed-viscosity case of the one-cell-identity question is excluded; an exact pressure criterion and a periodic kinetic-source condition; finite monomial-periodic obstructions	one-cell identity: answered: yes for the fixed-viscosity case; the literal-erratum question, the sublinear-pressure question and literal (B): open with reductions	the literal-erratum proofs

Result	Readings it settles	File
Ordinary printed (C) is answered: yes for the checked port of the whole-space construction; positive-time smoothness, the whole-space energy inequality and a zero-integral force variant extend it; a displayed common-force family is impossible	the ordinary printed (C) and its variants; (C) with weak solutions	the weak-class (C) proofs and the formal checks for (C); Lean: the whole-space breakdown and its closure
With the corrected identity tested through time zero and no initial term, a nonzero compactly supported start with zero force has no weak solution, in any weak class with a distributional initial value	(C) with weak solutions, the erratum applied to all of space, zero-extended identity: answered: yes in both weak classes	the zero-extended identity proof
The construction's force has zero mean, so a periodic Leray-Hopf solution from rest has zero mean velocity and competes with it	OpenAI's construction, weak solutions, zero mean velocity with periodic velocity and repeating pressure: answered: no; the logical combination is also in Lean	the weak zero-mean proof
The zero-mean-force result removes one qualification of the one-cell identity; a fixed force has a periodic solution at large viscosity	one-cell identity: zero-mean force answered: yes, one force for every viscosity answered: no, pressure gradient vanishing on the cell open	the one-cell-identity qualifications
The existing whole-space proofs cover the usual energy inequality, the energy bound with a pressure norm added, and a force of zero integral	the ordinary printed (C) variants: answered: yes for three readings of (7); energy finite at each time only, and a divergence-free force, open	the printed (C) variants

6. What remains open, exactly

- The literal-erratum question (chapter 6). Is there a C^∞ pair (u, p) on $R^3 \times [0, \infty)$ satisfying (1), (2), $u(\cdot, 0) = 0$ and $p(x + e_j, t) = p(x, t)$, with no periodicity or growth condition on u, for OpenAI's force? Any such u is unbounded relative to $1 + |x|$ on some $[0, T]$ with $T < 1$ (the superlinear-growth theorem of chapter 6), and carries a nonzero pressure-free period defect (the nonzero-defect lemma of chapter 6). Related: the free-velocity readings with $\nabla p = 0$ or with a pressure that is always periodic, and the one-cell-identity question below.
- The classical questions. (A) and (B) themselves, and (D) for the printed wording at non-rest starts. These are the classical questions (the classical (A) question, the classical (B) question, the remaining readings

of (D)). Literal (B), with velocity unrestricted, is implied by printed (B).

- The zero-mean-velocity question. Is there a competitor for OpenAI's force with zero mean velocity for all t? For periodic velocity at the matching viscosity the answer is no (3,440,640 readings: Lean for unit period and smoothness up to time zero, using that the construction's force has zero mean; the coarser-period and positive-time parts are written). For one force at every viscosity it is yes (2,580,480 readings, so the construction is answered: no). With weak solutions and no $\nabla p = 0$ requirement it is also yes (645,120 readings, so the construction is answered: no): a periodic Leray-Hopf solution from rest competes. The remaining 3,268,608 readings, with free velocity or with $\nabla p = 0$, are open: the cell boundary flux is not controlled.

- The cell-energy-inequality question. Is there a competitor for OpenAI's force that obeys the cell energy inequality and has a tilted pressure (periodic velocity), or a non-periodic velocity? Periodic velocity at the matching viscosity: no (3,440,640 readings). One force at every viscosity: yes (2,580,480 readings). The remaining 3,268,608 readings are open for the same boundary-flux reason.
- The sublinear-pressure question. The literal-erratum class with "pressure sublinear in x" in place of "pressure periodic". Still open: an exact pressure primitive criterion and a finite monomial-periodic obstruction are written, but a general kinetic source periodicity is not established.

- The one-cell-identity question. The literal-erratum class with the identity imposed on spatially periodic tests on one cell, in place of periodic pressure. For a fixed viscosity and the force conventions none, divergence-free projection or compact support it is excluded (2,064,384 readings, written). It is also excluded when the force must have zero mean, because the construction's own force has zero mean (688,128 readings, written, using the Lean zero-mean-force theorem). With one force for all viscosities and the corrected identity binding in full it is

answered: no (688,128 readings, written: a periodic solution exists at large viscosity). The remaining 688,128 readings (one force for all viscosities, with the pressure gradient required to vanish on the cell) are open.

- One force for all viscosities. For the ordinary periodic-pressure class it is answered: no (6,881,280 readings: a fixed force has a periodic-pressure solution at large viscosity). With $\nabla p = 0$ it is open (9,289,728 readings): excluding periodic forced Burgers competitors at every viscosity for one force.

- Groups decided in part. (D) with weak solutions and a remaining degenerate identity (answered: yes) and (B), the same readings (answered: no), are closed in the trace weak class; (A) with the pressure repeating at time zero only and smoothness only afterwards is answered: no; (D) with a start of minimal period one is answered: yes for 13,148,160 of its 21,934,080 readings; (A) with weak solutions and the erratum applied to all of space is answered: no and the corresponding (C) group answered: yes for the sub-readings of section 4, with 2,359,296 open readings for each of (A) and (C) (cell identities and pressure only at zero). OpenAI's construction with weak solutions and pressure freedom is open for 9,289,728 readings; its zero-mean part (645,120 readings) is answered: no. For the ordinary printed (C), energy finite at each time only (19,525,632 readings) and a divergence-free force (26,034,176 readings) are open, each for one missing step stated in the printed (C) variants. Each of these departs from the text in two places at once.

- The zero-mean force is closed: the construction's force has zero mean, so the zero-mean-force convention is answered: yes.

7. Further questions about the wording, and where each is handled

The points below have been raised about the wording. Each is given under its topic, with where it is handled. Text = settled by the official wording (section 1). Book = settled by a file already in the book. Matrix = given a verdict in

section 4. Open = section 6. Remaining = needs something other than writing.

- Chapter 1's plain table. Book: it says "divergence-free" and "decaying". Text: (4) is quoted in the statements page, Part 1.
- Printed and periodic-pressure existence for (B). Book and Matrix: the Lean equivalence; the equivalence lemma for (B).
- The mean of the (B) start. Matrix: the mean-of-the-start clause; the equivalence lemma for (B).
- The literal (B) reading. Matrix: literal (B) with the flow free, and the literal-erratum question. Open: the literal-erratum question and the classical questions. Book: chapter 2 and the check guide give the literal reading's coverage.
- Quantifier order in (D). Text: "Take $\nu > 0$ . . . Then there exist". Matrix: the viscosity clause. Corollary 10.6 states breakdown at $t = 1$ for every ν.
- Statement versus OpenAI's construction. Matrix: the (D) statement and the OpenAI column are separate tables.
- The residual in the Lean statement. Book: the residual is quoted in the statements page, Part 2; its terms are (1) with every term on the left.
- Where (1) holds. Matrix: that clause is irrelevant to every verdict.

- Extra force conventions (divergence-free, zero mean, compact support). Matrix: the force-convention clause; the divergence-free force reduction; the zero-mean-velocity question. Compact support uses OpenAI's statement.
- Test functions. Text: tests are compactly supported in $R^3 \times (0, \infty)$. No trace term and no extension enter. Book: chapter 6 and the answer key use no zero-extension convention.
- Conditions (4), (5), (7). Text: quoted. Matrix: the clauses for (4) and (5) never change a verdict. The clause for (7) changes only the ordinary printed (C): three of its readings give answered: yes, and energy finite at each time only stays open (section 3).
- Smoothness at $t = 0$. Book: smooth means $C\hat{}\infty$ on the closed half-space (Lean). Cited: Seeley. Matrix: the smoothness clause; the positive-time

exclusion and tilt removal.

- Extra conditions on the solution (energy, growth, mean). Matrix: the extra-conditions clause; the whole-space energy exclusion, the pressure-periodicity lemma and the zero-mean case of the witness forces; Open: the zero-mean-velocity and cell-energy-inequality questions.

- A solution on all of $\mathrm{R}^3 \times [0, \infty)$. Text: one solution "on $\mathrm{R}^3 \times [0, \infty)$". A compatible family of slab solutions is such a solution. No uniform-in-time bound is asked for.
- Periods. Matrix: the periods clause; the coarser-period reduction. Unit shifts give all integer shifts algebraically.
- Force bounds and the Euclidean norm. Book: the Lean equality of the two force classes supplies the direction that was missing (it is in the axiom list). Euclidean versus componentwise norm: a Lean inequality.
- The pressure sentence. Matrix: the pressure-sentence clause; the pressure-periodicity lemma and tilt removal. Book: chapter 2's gauge sentence.
- Whether (12) binds. Text: (12) is prose and is not listed in any alternative. Matrix: the identity clause; the extra-identity reduction, the frozen-vorticity obstruction, the shear-start obstruction and the curl of OpenAI's force.
- Weak solutions. Matrix: the smoothness clause (weak); cited: Leray, Hopf.
- A combination matrix. Matrix: this chapter is the combination matrix.

- Applying the erratum to all of space. Text: the erratum names (8) and (10) only. Matrix: that clause; the pressure-on-all-of-space lemma.
- Internal interfaces of the construction. Not needed for any verdict. Every verdict uses the slide-through escape, the forced breakdown with repeating pressure, the escape-pressure criterion, the failure of the escape under repeating pressure, the bridges named above, or written lemmas. The internal interfaces matter only through the statement of the forced breakdown, which the kernels check.
- Where the Lean definitions live. Book: the definitions for the printed problem and its pressure solution, and the Lean statement file with the

escape conclusion (quoted in the statements page), are in the Lean folder.
- The literal-erratum Lean sources. Book: they are in the Lean folder for the literal-erratum lemmas, in the separate build target; the default build is unchanged.
- Five points about the OpenAI paper and the build. Book: chapter 3 (both periods, each claim qualified by reading), chapter 2 (gauge), chapter 7 (evidence kinds).

- Reproducibility. Book: the Lean folder has the toolchain file, the lock file and the build file. The axiom printout lists only the three standard axioms on all nine lines (the check guide).
- Breakdown at $t = 1$ for each ν. Text: Corollary 10.6 states it. The forced breakdown with repeating pressure is existential and the book does not claim more.
- The slab quantifier. Book: chapter 6 states it.
- Superlinear growth. Book: chapter 6 derives superlinear growth from the linear-growth repetition theorem and calls the exponential bound an upper bound.
- Eleven failed constructions. Remaining: they are listed in chapter 6. Their working files are not in the book; they are available on request.

Credit

We stand on the shoulders of giants.

- Charles L. Fefferman wrote the problem statement, including the page-3 sentence that fixes the test functions. The Errata page is hosted by the Clay Mathematics Institute with it.
- The OpenAI team built the construction: their force, Corollary 10.6, the periodic uniqueness step, their candidate properties and the NavierStokesAndEuler package (Apache-2.0). It answers (C), and (D) under the erratum as intended.
- Sonny (2026) ported OpenAI's Lean package and checked its statements against Clay's clauses, tested it against every reading of

section 4 (the table for OpenAI's construction), extended the ordinary printed (C) to the variants listed there, proved the superlinear-growth theorem of chapter 6 and refuted Tao's Conjecture 1.8 (Lean).
- Sonny (2026) proved the other results of this chapter without OpenAI's construction: the slide-through escape, the equivalence for (B), the lemmas of section 5 other than the curl of OpenAI's force, the witness forces, and the verdicts of the (A), (B) and (D)-statement tables that do not cite the construction.

- The slide mechanism behind the slide-through escape, the sliding solution that a non-periodic pressure allows, is Terence Tao's (2011); he also observed the equivalence behind the remaining readings of (D). Applying the slide from rest to every push in Clay's class, the complete all-time proof and the Lean check are Sonny's (2026). The sliding solution is also in Koch, Nadirashvili, Seregin and Šverák (2009), and the sliding-frame symmetry in Fushchych and Popovych (1994).
- A. N. Tychonoff (1935) showed that uniqueness needs growth classes, which is why the literal-erratum question is hard.
- R. T. Seeley (1964), J. Leray (1934) and E. Hopf (1951) supplied the cited theorems.
- The kernel checkers are nanoda and lean4lean.

Check it yourself

the check guide has the recipe to recount the readings from the decision lists, and the reading splits describe each split in plain words.

Where to go next

- What is the one sentence about pressure that starts it all? See The updated question: the erratum.
- What exactly is left open for the periodic box? See What remains open.

11 Row splits

Row splits

Sonny. Some groups of readings in chapter 8 mix verdicts; this note splits them into sub-readings. The counts are counts of reading combinations; they give every reading equal weight and say nothing about which readings are reasonable. The splits, and the count of (A) under the identity readings, each have their own section near the end of this note. Each one is described in plain words in its section, so that anyone can write their own recount from the description (the recipe is in the check guide). Groups are described by the readings they stand for.

The proofs named below are in the closure proofs, the OpenAI-construction proofs, the literal-erratum proofs, the reading limits for (C) and the weak-class (C) proofs.

Sub-readings used in the tallies

Column	Sub-reading	Verdict	Readings	Basis
OpenAI's construction	one force for every viscosity, ordinary periodic pressure	Answered: no	6,881,280	the large-viscosity existence proof (the OpenAI-construction proofs)
OpenAI's construction	one force for every viscosity, with $\nabla p = 0$	Open	9,289,728	the uniform-viscosity reduction; the Lean uniform-viscosity exclusion equivalence
OpenAI's construction	the zero-mean force	Answered: yes	8,085,504	the Lean zero-mean-force theorem; the force-mean proofs
OpenAI's construction	zero mean velocity, periodic flow	Answered: yes	3,440,640	the Lean zero-mean-velocity exclusion; the cell-energy-inequality and coarser-period proofs
OpenAI's construction	zero mean velocity, one force for all ν, ordinary periodic pressure	Answered: no	2,580,480	the large-viscosity existence proof
OpenAI's construction	zero mean velocity, otherwise	Open	3,268,608	cell boundary flux not controlled
OpenAI's construction	cell energy inequality, periodic flow	Answered: yes	3,440,640	the cell-energy-inequality and coarser-period proofs
OpenAI's construction	cell energy inequality, one force for all ν, ordinary periodic pressure	Answered: no	2,580,480	the large-viscosity existence proof
OpenAI's construction	cell energy inequality, otherwise	Open	3,268,608	cell boundary flux not controlled
OpenAI's construction	one-cell identity, fixed viscosity	Answered: yes	2,064,384	the literal-erratum proofs, the one-cell boundary, derivative-matching and one-cell exclusion theorems
OpenAI's construction	one-cell identity, the force must have zero mean (viscosity for each force or fixed)	Answered: yes	688,128	the one-cell-identity qualifications, part 1; the zero-mean-force result
OpenAI's construction	one-cell identity, one force for every viscosity, no condition on the	Answered: no	688,128	the one-cell-identity qualifications, part 2; the

Column	Sub-reading	Verdict	Readings	Basis
	pressure gradient			large-viscosity existence proof
OpenAI's construction	one-cell identity, one force for every viscosity, pressure gradient required to vanish on the cell	Open	688,128	the one-cell-identity qualifications, part 3; the pressure-free forced Burgers question
OpenAI's construction	weak solutions, zero mean velocity, periodic velocity and repeating pressure, no condition on the pressure gradient	Answered: no	645,120	the weak zero-mean proof; the other weak pressure-freedom readings, 9,289,728 in all, stay open
(D)	weak solutions with a remaining degenerate identity	Answered: yes	12,902,400	the closure proofs, section 3
(D)	start of minimal period one, degenerate part and periodic flow with strong pressure	Answered: yes	13,148,160	the closure proofs, sections 3 and 7
(D)	start of minimal period one, otherwise	Open	8,785,920	the closure proofs, section 7
(B)	weak solutions with a remaining degenerate identity	Answered: no	11,354,112	the closure proofs, section 3
(A)	weak solutions, erratum applied to all of space, degenerate or decaying-swirl cases	Answered: no	35,684,352	the closure proofs, sections 2 and 4
(A)	weak solutions, erratum applied to all of space, cell identity or pressure at zero	Open	2,359,296	the closure proofs, section 5
(A)	pressure at zero only, smooth afterwards	Answered: no	1,769,472	the closure proofs, section 8
(A)	identity imposed on (A) itself, erratum in the box only, the text's test fields, smooth up to time zero	Answered: no	16,515,072	Lean, nanoda: Clay's 2000 wording stated directly (chapter 8, section 5)
(C)	weak solutions, erratum applied to all of space, degenerate or periodic positive-time pressure Laplacian	Answered: yes	26,050,560	the closure proofs, section 4; the weak-class (C) proofs

Column	Sub-reading	Verdict	Readings	Basis
(C)	weak solutions, erratum applied to all of space, corrected identity tested through time zero with zero extension	Answered: yes	9,633,792	the zero-extended identity proof
(C)	weak solutions, erratum applied to all of space, otherwise	Open	2,359,296	the cell identity and pressure-at-zero sub-readings, as for (A)
(C)	the ordinary printed (C), one common force for every viscosity	Open as a group (answered: no if the start must also be common; open if the start may depend on the viscosity)	52,068,352	the common-force note
(C)	the ordinary printed (C), condition (7) read as one bound for all times, the usual energy inequality, or the energy bound with a pressure norm added; force convention none or compact support in time	Answered: yes	39,051,264	the Lean whole-space breakdown; the positive-time smoothness proof; the printed (C) variants
(C)	the ordinary printed (C), the force must have zero integral over space, same three readings of (7)	Answered: yes	19,525,632	the zero-integral force proof; the printed (C) variants
(C)	the ordinary printed (C) variants, otherwise: energy finite at each time only, or a divergence-free force required	Open	45,559,808	the printed (C) variants, settlements 4 and 5

One choice is made in the counts and should be read as a choice, not as a result:

- Two meanings in the ordinary printed (C). The readings "pressure norm added to the energy bound" and "zero integral of the force" are counted answered: yes on the meanings stated in the printed (C) variants: the printed energy bound is kept, and the integral is over all of space. If the pressure norm replaced the energy bound, 19,525,632 readings would return to open. If the zero integral were taken over one unit cell, the

19,525,632 zero-integral readings would be open, since a zero integral over one cell is not proved.

The finer split at the close of the work

Counts in this table are computed counts of the chapter 8 classifier. They describe which source combinations reach each subclass. They are not counts of separate fluid theorems or a measure of which interpretation is reasonable. The trace weak class, the distinction between a pressure value and a trace, and the Leray-Hopf refinements are defined in the closure proofs, sections 1 and 5. An additional analytic convention is not silently counted as an extra clause in the original product.

In the table, "the pressure-free identity" is the pressure-free compatibility equation and "the frozen-curl identity" is the frozen-curl compatibility equation of the closure proofs, section 2; "the pressure-free Burgers system" is the system of its section 6.

Source group	Subclass	Verdict and scope	Basis	Count
(D), weak, remaining degenerate identities	pressure-free identity, globally or on a cell	Answered: yes in the trace weak class	Lean and nanoda for the identity on the text's test fields; written for divergence-free-only and one-cell tests, section 3, pressure-free compatibility	9,461,760
(D), weak, remaining degenerate identities	frozen-curl identities with a primitive start	Answered: yes in the trace weak class	Lean and nanoda for the identity on the text's test fields; written for divergence-free-only and one-cell tests, section 3, frozen-curl compatibility	3,440,640
(B), weak, remaining degenerate identities	pressure-free identity, globally or on a cell	Answered: no in the trace weak class	Lean and nanoda for the identity on the text's test fields; written for divergence-free-only and one-cell tests, section 3, pressure-free compatibility	7,569,408
(B), weak, remaining degenerate identities	remaining frozen-curl identities	Answered: no in the trace weak class	Lean and nanoda for the identity on the text's test fields; written for divergence-free-only and one-cell tests, section 3, frozen-curl compatibility	3,784,704
(A), pressure at zero only, smooth afterwards	both fields continuous at zero, periodic pressure trace	Answered: no	Written, section 8	1,769,472
(A), weak, erratum applied to all of space	pressure-free degeneracy	Answered: no in the trace weak class	Lean and nanoda for the identity on the text's test fields; written for divergence-free-only and one-cell tests, section 4	5,505,024
(A), weak, erratum applied to all of space	frozen-curl degeneracy	Answered: no in the trace weak class	Lean and nanoda for the identity on the text's test fields; written for divergence-free-only and one-cell tests, section 4	16,515,072

Source group	Subclass	Verdict and scope	Basis	Count
(A), weak, erratum applied to all of space	binding zero-extended identity	Answered: no in the trace weak class	Lean and nanoda; written, sections 2 and 4	9,633,792
(A), weak, erratum applied to all of space	positive-time periodic pressure Laplacian	Answered: no in the trace weak class with whole-space finite energy	Lean and nanoda when the pressure itself repeats at every positive time and the energy bound is uniform in time; written otherwise, section 4	4,030,464
(A), weak, erratum applied to all of space	nondegenerate cell identity	still open in the trace weak class; answered: no in the Leray-Hopf refinement	Exact torus-extension reduction and local smooth comparison, section 5	786,432
(A), weak, erratum applied to all of space	pressure only at zero, no remaining binding identity	Answered: yes for an assigned value and the usual weak energy readings; answered: no for a genuine pressure trace; an unspecified pressure norm remains open	Section 5, cited weak existence and written trace obstruction	1,572,864
(C), weak, erratum applied to all of space	pressure-free degeneracy	Answered: yes in the trace weak class	Lean and nanoda for the identity on the text's test fields; written for divergence-free-only and one-cell tests, section 4	5,505,024
(C), weak, erratum applied to all of space	frozen-curl degeneracy	Answered: yes in the trace weak class	Lean and nanoda for the identity on the text's test fields; written for divergence-free-only and one-cell tests, section 4	16,515,072
(C), weak, erratum applied to all of space	binding zero-extended identity	Answered: yes in both meanings of "weak solution" (the trace weak class and the usual weak class)	Lean and nanoda (in the weak class of the zero-extended identity proof, and against the trace weak class); written, the zero-extended identity proof; the closure proofs cover the trace weak class	9,633,792
(C), weak, erratum applied to all of space	positive-time periodic pressure Laplacian	Answered: yes in the trace weak class with whole-space finite energy	Lean and nanoda when the pressure itself repeats at every positive time and the energy	4,030,464

Source group	Subclass	Verdict and scope	Basis	Count
			bound is uniform in time; written otherwise, section 4	
(C), weak, erratum applied to all of space	nondegenerate cell identity	still open in the trace weak class; answered: yes in the Leray-Hopf refinement	Exact torus-extension reduction and local smooth comparison, section 5	786,432
(C), weak, erratum applied to all of space	pressure only at zero, no remaining binding identity	Answered: no for an assigned value and the usual weak energy readings; answered: yes for a genuine pressure trace; an unspecified pressure norm remains open	Section 5, cited weak existence and written trace obstruction	1,572,864
OpenAI's construction, weak, pressure freedom	global pressure freedom, periodic velocity, data for each viscosity	Answered: yes in the Leray-Hopf refinement; still open in the trace weak class without its energy inequality	Written constrained Burgers comparison, section 6	2,322,432
OpenAI's construction, weak, pressure freedom	same, one chosen viscosity	Answered: yes in the Leray-Hopf refinement; still open in the trace weak class without its energy inequality	Section 6	2,322,432
OpenAI's construction, weak, pressure freedom	same, one force for all viscosities	still open: the family/admissibility quantifier remains in addition to the weak-class issue	Section 6	2,322,432
OpenAI's construction, weak, pressure freedom	global pressure freedom, unrestricted velocity, data for each viscosity	still open; exactly the pressure-free Burgers system with the retained extra conditions	Section 6	516,096
OpenAI's construction, weak, pressure freedom	same, one chosen viscosity	still open, same exact reduction	Section 6	516,096
OpenAI's construction, weak, pressure freedom	same, one force for all viscosities	still open, same reduction with its uniform force quantifier	Section 6	516,096
OpenAI's construction,	pressure freedom on a cell only, data for	still open; the local pressure-free Burgers system, exterior	Section 6	258,048

Source group	Subclass	Verdict and scope	Basis	Count
weak, pressure freedom	each viscosity	momentum and cell identity all retained		
OpenAI's construction, weak, pressure freedom	same, one chosen viscosity	still open, same exact reduction	Section 6	258,048
OpenAI's construction, weak, pressure freedom	same, one force for all viscosities	still open, same reduction with its uniform force quantifier	Section 6	258,048
OpenAI's construction, weak, pressure freedom	zero mean velocity and strong pressure, data for each viscosity	Answered: no (the construction's force has zero mean at every time)	Written mean balance plus cited weak existence, section 6, and the Lean zero-mean-force theorem; the weak zero-mean proof	215,040
OpenAI's construction, weak, pressure freedom	same, one chosen viscosity	Answered: no, same reason	the weak zero-mean proof	215,040
OpenAI's construction, weak, pressure freedom	same, one force for all viscosities	Answered: no: the construction's force has a periodic Leray-Hopf competitor at every viscosity	the weak zero-mean proof	215,040
(D), start of minimal period one	remaining degenerate identities with primitive start	Answered: yes, including a single zero force for every viscosity	Written sections 3 and 7	10,321,920
(D), start of minimal period one	periodic velocity and strong pressure, data for each viscosity	Answered: yes	Written preparation and the forced breakdown with repeating pressure, section 7	1,413,120
(D), start of minimal period one	same, one chosen viscosity	Answered: yes	Written preparation and the forced breakdown with repeating pressure, section 7	1,413,120
(D), start of minimal period one	same, one force for all viscosities	still open; the preparatory force depends on viscosity	Section 7	1,413,120

Source group	Subclass	Verdict and scope	Basis	Count
(D), start of minimal period one	periodic velocity, pressure controlled by a free tilt, data for each viscosity	still open; exact controlled mean-zero equation	Section 7	1,597,440
(D), start of minimal period one	same, one chosen viscosity	still open, same exact reduction	Section 7	1,597,440
(D), start of minimal period one	same, one force for all viscosities	still open, same reduction with its uniform force quantifier	Section 7	1,597,440
(D), start of minimal period one	unrestricted velocity, data for each viscosity	still open; periodic subclasses excluded by preparation; unrestricted growth and defect system retained	Section 7	860,160
(D), start of minimal period one	same, one chosen viscosity	still open, same exact reduction	Section 7	860,160
(D), start of minimal period one	same, one force for all viscosities	still open, same reduction with its uniform force quantifier	Section 7	860,160

Within each source group these counts sum to that group's chapter 8 count. For the minimal-period (D) group the written closures cover 13,148,160 source combinations and the open residual subclasses cover 8,785,920. In each of the (A) and (C) groups with weak solutions and the erratum applied to all of space, 35,684,352 source combinations have the robust obstruction proved for the trace weak class. The remaining two buckets are displayed with their precise class and trace distinctions.

The viscosity phrase "one force for all ν" does not state whether the start is also required to be common. The zero-force primitive-start obstruction is uniform in both objects, so it handles either convention. The remaining uniform-force subclasses are left open under both the interpretation allowing a viscosity-dependent start and the stronger interpretation requiring a common start. No quantifier is interchanged in either direction.

The counterfactual variant of (A) with the pressure at zero only, in which only velocity is continuous and pressure has merely an assigned zero-time value, is not part of the quoted pair-continuity group. Its exact residual equivalence is given in the written proof, section 8 of the closure proofs, without changing the chapter count.

The split of the five groups of OpenAI's construction

Counts below retain membership in the original groups of chapter 8; they are not a renumbered decision list. Each original reading combination has the same weight as in the chapter. These weights do not measure plausibility or significance.

Verdict counts within the five groups

Original group	Original count	Answered: yes	Answered: no	Still open
One force for every viscosity	16,171,008	0	6,881,280	9,289,728
The zero-mean force	8,085,504	8,085,504	0	0
Zero mean velocity	9,289,728	3,440,640	2,580,480	3,268,608
The cell energy inequality	9,289,728	3,440,640	2,580,480	3,268,608
Unrestricted velocity, pressure-free	4,128,768	0	0	4,128,768
Group total	46,964,736	14,966,784	12,042,240	19,955,712

The counts are computed from the chapter's unchanged decision code. The mathematical bases are the force-mean theorem, the written proofs, and the open gates below. No change is made to the classical (A) question, the classical (B) question or the remaining readings of (D), and no dependent recount of their column is claimed.

The zero-mean force

Answered: yes. The Lean zero-mean-force theorem strengthens the actual construction witness by giving the exact zero vector for its centered unit-cell integral, at every real time, while retaining the candidate properties and periodic-pressure exclusion at the matching positive viscosity. A companion Lean result proves the corresponding whole-space integral before

periodization. The source's force formula is reconstructed from the selected raw stages, then passed through the original compact cutoff, positive-time cutoff, viscosity scaling, delayed compression and lattice periodization.

This is a statement about the explicit construction formula. OpenAI's candidate properties by themselves do not fix a force's continuation after time one, and they are not being used as an all-time mean theorem. The proof retains this distinction. The force may be chosen for each viscosity, exactly as in the forced breakdown with repeating pressure.

One force for every viscosity

The original group contains 6,881,280 ordinary periodic-pressure combinations and 9,289,728 combinations requiring grad p=0. These are different existence questions.

For the ordinary class, answered: no: a fixed smooth periodic force admits a global smooth periodic-pressure solution from rest when the viscosity is sufficiently large. The large-viscosity existence proof gives the Fourier estimates, barrier, higher-order bounds and limiting construction. The actual zero-mean force then gives a zero-mean solution as well, and its compact time support gives bounded cell energy and a smooth zero extension near time zero whenever a reading requires those. The same solution obeys the cell energy equality. Thus the extra conventions present in this ordinary slice do not defeat this competitor.

For grad p=0, still open for a single force from this construction at every viscosity. The large-viscosity argument recovers a pressure whose gradient may be nonzero. The forced breakdown with repeating pressure excludes the pressure-free periodic competitor at the viscosity for which the force is constructed. It does not quantify over every other viscosity for that same force. The remaining condition is exclusion of periodic incompressible forced Burgers competitors at every viscosity for that fixed force. A Lean result (the uniform-viscosity exclusion equivalence) checks this exact equivalence, with the force outside the viscosity quantifier on both sides. The uniform-viscosity

reduction explains it. The construction supplies exclusion at its matching viscosity and leaves the universal condition open.

The statement that a supplied force is chosen per viscosity is not, by itself, a disproof of a uniform-force statement. The ordinary slice is decided by the separate existence proof.

Zero mean velocity

For periodic velocity at a fixed or matching viscosity, answered: yes: 3,440,640 combinations. Zero mean force and zero mean velocity give zero pressure tilt. The forced breakdown with repeating pressure then excludes the competitor. The cell-energy-inequality proof gives the argument, including the positive-time-only smoothness extension by continuity at the start. The coarser-period proof reduces coarser integer velocity periods to unit periods. The force's mean-zero premise is Lean. A Lean result (the zero-mean-velocity exclusion) also checks the full exclusion for unit-period velocity with smoothness up to time zero. The coarser-period proof covers integer-coarser periods, and the continuity argument in the cell-energy-inequality proof covers positive-time-only smoothness.

For one force at all viscosities, the ordinary periodic-pressure solution of the large-viscosity existence proof gives a zero-mean competitor at a sufficiently large viscosity. This decides 1,720,320 periodic-velocity combinations as answered: no. It also decides 860,160 unrestricted-velocity combinations as answered: no: they have no grad p=0 requirement, so a periodic competitor is allowed. The latter count comprises 688,128 with no additional identity and 172,032 with the corrected one-cell identity.

The other 3,268,608 unrestricted-velocity combinations remain open. The chapter's mean-balance argument expressly concerns the printed class. Its group also contains free velocities, for which cell boundary fluxes need not cancel. The zero-cell-mean example gives a zero-force, zero-cell-mean, pressure-free nonperiodic flow from rest, showing that zero mean over one cell alone does not imply periodicity. That example is not a competitor for the assigned nonzero force.

For unrestricted velocity, a spatial mean also needs a specified averaging convention. The one-cell statements here use the fixed centered unit cell. No theorem about an unspecified spatial-infinity average is asserted.

The cell energy inequality

For periodic velocity at a fixed or matching viscosity, answered: yes: 3,440,640 combinations. The cell-energy-inequality proof compares the exact smooth energy identity with the ordinary forced cell energy inequality. The difference is one half of the squared mean velocity, forcing both mean velocity and pressure tilt to vanish. Periodic uniqueness and the candidate blowup then exclude a global competitor. Only zero force mean before time one is needed for this exclusion. The coarser-period proof covers coarser integer periods before the unit-cell argument is applied. A Lean energy-mean module checks the algebraic last step and the wrong-sign counterexample; the PDE integration remains written.

The same 2,580,480 uniform-viscosity combinations described for zero mean velocity are answered: no by the large-viscosity existence proof, since its periodic-pressure solution satisfies the energy equality.

The remaining 3,268,608 unrestricted-velocity combinations are open. For a free velocity on a fixed cell Q, the exact energy identity contains the boundary flux

integral_boundary_Q [(|u|^2/2+p)u - nu grad(|u|^2/2)].n

The boundary terms vanish for periodic traces. They need not vanish for free traces. The cell inequality supplies a constraint on this flux, but the present proof does not turn that constraint into periodicity or a growth bound at spatial infinity. Requiring the inequality on every translated cell would be an additional precise convention requiring its own argument.

Unrestricted velocity, grad p=0

Still open: 4,128,768 combinations. At the matching viscosity, the forced breakdown with repeating pressure excludes periodic velocity. A free-velocity

competitor would have to solve the forced viscous Burgers equation, remain divergence-free, and satisfy

div f = sum_(i,j) (partial_i u_j)(partial_j u_i).

The pressure-free compatibility identity derives this identity. A Lean result checks the full existence equivalence for smoothness up to zero, and another excludes the periodic subclass. Neither result decides unrestricted existence. A Lean initial-gap result also checks that the construction pipeline permits a positive initial interval on which the force is zero, so finite initial-jet tests cannot decide this gate. The zero-force flow of the zero-cell-mean example shows why dropping growth conditions cannot be replaced by an unsupported uniqueness assertion. It is not a solution for the actual force. The chapter 6 growth and defect exclusions remain conditional restrictions on a possible competitor.

Splits on top of the decision lists

The splits below decide the groups listed and nothing else: every other row keeps its count, each column still sums to 440,401,920, and there are 82 distinct patterns of verdicts. How the 82 is counted: for each of the 3,440,640 enumerated combinations, the pattern is the list of its five verdicts, one per column; where a column's verdict depends on condition (7), that entry is the set of its verdicts. 82 different lists occur. So that anyone can write their own recount, here is each split in plain words, with the weight of the readings it covers. All other rules, and the way the four clauses that never change a verdict are counted (chapter 8, section 3), are as in the decision lists of chapter 8.

Split	Readings it covers	Verdict without this split	Verdict with this split	Readings	Proof
(C), weak solutions, erratum applied to all of space	the corrected identity is tested through time zero with zero extension and no initial term	Open	Answered: yes	9,633,792	the zero-extended identity proof
OpenAI's construction, weak solutions	the rows left open only for "periodic velocity, repeating pressure, zero mean velocity" and not for "pressure gradient zero"	Open	Answered: no	645,120	the weak zero-mean proof
OpenAI's construction, velocity free, corrected identity binding on one cell	viscosity for each force or fixed, force must have zero mean	Open	Answered: yes	688,128	the one-cell-identity qualifications, part 1
the same group	one force for every viscosity, the full corrected identity binding (no condition that the pressure gradient vanishes)	Open	Answered: no	688,128	the same, part 2
the same group	one force for every viscosity, only the time sign of the identity corrected (the pressure gradient vanishes on the cell)	Open	Open	688,128	the same, part 3
(D) as a statement	readings that reach "OpenAI's construction meets the reading" and are now answered: yes in the rows above	Open	Answered: yes	258,048	the OpenAI-column rule of chapter 8, section 4
(C), ordinary printed wording, force convention none or compact support in time	condition (7) read as the usual energy inequality	Open	Answered: yes	13,017,088	the printed (C) variants
the same	condition (7) read as the energy bound with a pressure norm added	Open	Answered: yes	13,017,088	the same

Split	Readings it covers	Verdict without this split	Verdict with this split	Readings	Proof
(C), ordinary printed wording, force must have zero integral over space	the three readings of (7) that keep an energy bound	Open	Answered: yes	19,525,632	the same

The ordinary printed (C) readings with energy finite at each time only (19,525,632) and with a divergence-free force (26,034,176) stay open. The readings with one common force for every viscosity (52,068,352) are open as a group; see the note below.

- The ordinary printed (C), one common force for every viscosity (52,068,352 readings, open as a group). The phrase fixes the force but not the start, and the reading matrix has no clause for the start, so the group has two meanings that the counts cannot separate. With one start and one force, the readings are answered: no: by scaling, a fixed pair of start and force becomes small data at large viscosity, and the classical small-data theory gives a global solution (cited; the reduction is written in the common-force note; the logic is checked in Lean there). If the start may depend on the viscosity, nothing is known: answering no would answer the classical question (A) yes, which the book cannot do (Lean reduction in the same note), and answering yes would need a force that defeats some start at every viscosity at once, which OpenAI's construction does not give (its force changes with the viscosity). The reading limits for (C), item 1, shows that OpenAI's displayed scaled family is not one common force (written and Lean). Further research is recommended.

The count of (A) under the identity readings

This row answers no for one group of (A) readings that would otherwise fall to the last, open row, and decides nothing else: every other row keeps its count and each column still sums to 440,401,920. It follows a Lean result stated directly against Clay's 2000 wording (nanoda accepted an export of

93,579 declarations): with a degenerate or binding identity imposed on (A) itself, some decaying start has no solution.

Split	Readings it covers	Verdict without this split	Verdict with this split	Readings	Proof
(A), erratum applied to the box only	the identity clause read as printed (12) also binding, only the time sign corrected, or only the pressure sign corrected; the text's test functions; smoothness C^∞ up to t = 0, or the restriction of a C^∞ function on R^3 × R	Open	Answered: no	16,515,072	Lean, nanoda (chapter 8, section 5, Clay's 2000 wording stated directly); the restriction reading is a subclass of smoothness up to t = 0

To recount: in the (A) decision list of chapter 8, put this row just before the last row (the classical (A) question). It takes 3 identity readings × 1 test reading × 2 smoothness readings × the $7 \cdot 4 \cdot 4 \cdot 4 \cdot 3 \cdot 8 \cdot 2$ readings of the pressure sentence, the label, the force conventions, the periods, the viscosity, the extra conditions and the mean of the start, which is 129,024 of the 3,440,640 combinations, times 128. Not covered by this row: the same identities with smoothness for $t > 0$ only, with weak solutions (already decided by the weak rows), and with tests that are divergence-free only, reach $t = 0$ with zero extension, or live on one cell; the erratum applied to all of space is decided by its own rows. The smooth zero extension, which the same Lean files also treat, is answered: no by its own row.

With this row, (A) is answered: yes in 11.4%, answered: no in 57.3% and open in 31.3% of all readings; the classical-question row holds 135,659,520 readings.

Evidence scope

Lean means the stated Lean declarations compiled under the pinned toolchain with printed transitive axioms limited to propext, Classical.choice and Quot.sound. Cached imported libraries and inherited project objects are used; these checks are not a clean rebuild of every upstream source and are not a separate-kernel check.

Written refers to the complete arguments in the OpenAI-construction proofs. The count enumeration is computed. The unrestricted-velocity existence gates and the pressure-free uniform-force gate are open. The group verdicts report an entire original group as still open when it contains both decided and open subreadings. The final import check prints the axioms of 58 named declarations from 17 accepted modules. All printed axioms are among propext, Classical.choice and Quot.sound. A Lean control with a different force shows that the zero-force-mean premise of the zero-mean-velocity question cannot simply be dropped. Source and object bindings and the exact printed output are kept in the supporting files, available on request.

The Lean names behind the results mentioned above are listed in the notes of the OpenAI-construction proofs.

Where to go next

- Where do these splits enter the counts? See Every reading, section 4.
- How do you recount the readings? See the check guide.

12 Exact statements

Exact statements

Part 1 quotes Clay's official problem description (C. L. Fefferman, Existence and smoothness of the Navier-Stokes equation, as hosted by the Clay Mathematics Institute, with its Errata page). Part 2 quotes the Lean statements word for word from the Lean sources in this book, with file and line.

Part 1. The official description

The copy quoted here is the PDF hosted by the Clay Mathematics Institute, SHA-256 c1b5f27b1a64705cfaf1afceea513db5deedca8a18ca56ab32e7f86445a06d0c. Mathematical symbols are transcribed.

Equations (1)-(3), page 1. "(1) $(\partial/\partial t)u_i + \Sigma_{j=1}^n u_j \partial u_i/\partial x_j = \nu \Delta u_i - \partial p/\partial x_i + f_i(x, t)$ $(x \in R^n, t \geq 0)$, (2) div $u = \Sigma_{i=1}^n \partial u_i/\partial x_i = 0$ $(x \in R^n, t \geq 0)$ with initial conditions (3) $u(x, 0) = u^\circ(x)$ $(x \in R^n)$. Here, $u^\circ(x)$ is a given, C^∞ divergence-free vector field on R^n, $f_i(x, t)$ are the components of a given, externally applied force (e.g. gravity), ν is a positive coefficient (the viscosity), and $\Delta = \Sigma_{i=1}^n \partial^2/\partial x_i^2$ is the Laplacian in the space variables."

Whole-space conditions (4)-(7), page 1. "For physically reasonable solutions, we want to make sure $u(x, t)$ does not grow large as $|x| \to \infty$. Hence, we will restrict attention to forces f and initial conditions u° that satisfy (4) $|\partial_x^\alpha u^\circ(x)| \leq C_{\alpha K} (1 + |x|)^{-K}$ on R^n, for any α and K and (5) $|\partial_x^\alpha \partial_t^m f(x, t)| \leq C_{\alpha m K} (1 + |x| + t)^{-K}$ on $R^n \times [0, \infty)$, for any α, m, K. We accept a solution of (1), (2), (3) as physically reasonable only if it satisfies (6) $p, u \in C^\infty(R^n \times [0, \infty))$ and (7) $\int_{R^n} |u(x, t)|^2 \, dx < C$ for all $t \geq 0$ (bounded energy). Alternatively, to rule out problems at infinity, we may look for spatially periodic solutions of (1), (2), (3). Thus, we assume that $u^\circ(x)$, $f(x, t)$ satisfy"

Periodic data (8). "$u^\circ(x + e_j) = u^\circ(x)$, $f(x + e_j, t) = f(x, t)$ for $1 \leq j \leq n$ (e_j = jth unit vector in R^n)."

Force decay (9). "In place of (4) and (5), we assume that u° is smooth and that (9) $|\partial_x^\alpha \partial_t^m f(x, t)| \leq C_{\alpha m K} (1 + |t|)^{-K}$ on $R^3 \times [0, \infty)$, for any α, m, K."

Solution conditions (10), (11). "We then accept a solution of (1), (2), (3) as physically reasonable if it satisfies (10) $u(x, t) = u(x + e_j, t)$ on $R^3 \times [0, \infty)$ for $1 \leq j \leq n$ and (11) $p, u \in C^\infty(R^n \times [0, \infty))$."

(A) Existence and smoothness of Navier-Stokes solutions on R^3. "Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x, t)$ to be identically zero. Then there exist smooth functions $p(x, t)$, $u_i(x, t)$ on $R^3 \times [0, \infty)$ that satisfy (1), (2), (3), (6), (7)."

(B) Existence and smoothness of Navier-Stokes solutions in R^3/Z^3. "Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (8); we take $f(x, t)$ to be identically zero. Then there exist smooth functions $p(x, t)$, $u_i(x, t)$ on $R^3 \times [0, \infty)$ that satisfy (1), (2), (3), (10), (11)."

(C) Breakdown of Navier-Stokes solutions on R^3. "Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on R^3 and a smooth $f(x, t)$ on $R^3 \times [0, \infty)$, satisfying (4), (5), for which there exist no solutions (p, u) of (1), (2), (3), (6), (7) on $R^3 \times [0, \infty)$."

(D) Breakdown of Navier-Stokes Solutions on R^3/Z^3. "Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on R^3 and a smooth $f(x, t)$ on $R^3 \times [0, \infty)$, satisfying (8), (9), for which there exist no solutions (p, u) of (1), (2), (3), (10), (11) on $R^3 \times [0, \infty)$."

Weak form (12), (13), page 3. "For instance, if (1) and (2) hold, then, for any smooth vector field $\theta(x, t) = (\theta_i(x, t))_{1\le i\le n}$ compactly supported in $R^3 \times (0, \infty)$, a formal integration by parts yields (12) $\iint_{R^3\times R} u \cdot \partial\theta/\partial t \, dx \, dt - \Sigma_{ij} \iint_{R^3\times R} u_i u_j \, \partial\theta_i/\partial x_j \, dx \, dt = \nu \iint_{R^3\times R} u \cdot \Delta\theta \, dx \, dt + \iint_{R^3\times R} f \cdot \theta \, dx \, dt - \iint_{R^3\times R} p \cdot (\mathrm{div}\ \theta) \, dx \, dt$. [...] Similarly, if $\varphi(x, t)$ is a smooth function, compactly supported in $R^3 \times (0, \infty)$, then a formal integration by parts and (2) imply (13) $\iint_{R^3\times R} u \cdot \nabla_x \varphi \, dx \, dt = 0$. A solution of (12), (13) is called a weak solution of the Navier-Stokes equations." None of (A)-(D) lists (12) or (13) among its conditions.

Errata (page 6). "The further condition $p(x + e_j, t) = p(x, t)$ should be made explicit in Eqn (8). Eqn (10) should read: $- \iint_{R^3\times R} u \cdot \partial\theta/\partial t \, dx \, dt - \Sigma_{ij} \iint_{R^3\times R} u_i u_j \, \partial\theta_i/\partial x_j \, dx \, dt = \nu \iint_{R^3\times R} u \cdot \triangle\theta \, dx \, dt + \iint_{R^3\times R} f \cdot \theta \, dx \, dt + \iint_{R^3\times R} p \cdot (\mathrm{div}\ \theta) \, dx \, dt$". This is (12) with the signs of its time and pressure terms reversed (chapter 2). Chapter 8 takes every reading of both sentences.

Part 2. The Lean statements

Each result below has the same layout. Clay's question says which alternative and which wording it speaks to. Answer says in plain words what the Lean

result says. The box, Lean code, for checking it yourself, is not Clay's and is not part of the problem: the Lean names are internal labels, not terms of the problem. The code is quoted word for word, with the file and line as a comment on its opening line. Paths in the comments are relative to the book's Lean folder.

In the code, "Periodic F" means that F repeats under a shift by one in each space direction, on $\mathrm{R}^3 \times [0, \infty)$; its definition is the opening block below. The definitions the printed solution class uses are these:

```
-- file: lean/JustCheckin/Statement.lean, line 36
-- Periodic F means: ∀ z ∈ domain, ∀ j : Fin 3, F (z.1 + EuclideanSpace.si
```

```
-- file: lean/JustCheckin/Statement.lean, line 58
structure PrintedSolution (ν : ℝ) (f u : Velocity) (p : Pressure) : Prop wh
  velocity_smooth : Smooth u
  pressure_smooth : Smooth p
  momentum : ∀ z ∈ domain, residual ν f u p z = 0
  incompressible : ∀ z ∈ domain, divergence u z = 0
  initial_rest : ∀ x : Space, u (x, 0) = 0
  velocity_periodic : Periodic u
```

```
-- file: lean/JustCheckin/Geometry.lean, lines 35 and 42
def domain : Set SpaceTime := univ ×ˢ Ici (0 : ℝ)

def Smooth {E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E]
    (F : SpaceTime → E) : Prop := ContDiffOn ℝ ∞ F domain
```

```
-- file: lean/JustCheckin/Statement.lean, lines 52 and 55
-- (derivative F z is fderivWithin ℝ F domain z)
def divergence (u : Velocity) (z : SpaceTime) : ℝ :=
  ∑ j : Fin 3, «partial» j u z j

def residual (ν : ℝ) (f u : Velocity) (p : Pressure) (z : SpaceTime) : Spa
  timePartial u z + derivative u z (u z, 0) - ν • laplacian u z + gradient
```

In words: the printed solution class has no pressure periodicity (it is the printed (10) and (11), from rest). Where a result needs the erratum's class, it adds that the pressure repeats. "Smooth" means C^∞ up to $t = 0$ on the closed half-space, as in (11), and the residual being zero is equation (1) with every term moved to the left. The momentum and incompressibility conditions hold for $t \geq 0$, $t = 0$ included. For fields that are C^∞ up to $t = 0$ this is the same as requiring them for $t > 0$, by continuity.

The conditional zero-force drift

Credit: Builds on: the observation of Terence Tao (2011). Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (D), the repeating box, with a force, under the original wording. This result is about the printed wording of (10) and (11), which does not make the pressure repeat. With no force and a fluid at rest, the printed wording allows both the still fluid and a drifting flow, provided one has a smooth path that starts with zero position and zero velocity and is not zero later. The drifting flow is conditional on that path. In symbols: for every $\nu > 0$ and every smooth path Y with $Y(0) = 0$, speed $Y'(0) = 0$ and $Y(t) \neq 0$ for some t in the time set,

$$\text{PrintedProblem}(\nu, 0, 0, 0, 0) \ \wedge \ \text{PrintedProblem}(\nu, 0, 0, \text{driftVelocity}\, Y, \text{driftPress}$$

> Lean code, for checking it yourself (the Lean names are internal labels, not terms of the problem). Checked by Lean, nanoda and lean4lean.

```
-- file: lean/JustExploring/Data.lean, line 55
theorem zero_data_nonuniqueness_in_printed_problem (v : ℝ) (hv : 0 < v)
    {Y : ℝ → Vec} (hY : SmoothPath Y) (hY0 : Y 0 = 0) (hv0 : speed Y 0 = 0)
    (hnonzero : ∃ t ∈ times, Y t ≠ 0) :
    PrintedProblem v (fun _ => 0) (fun _ => 0) (fun _ => 0) (fun _ => 0) ∧
    PrintedProblem v (fun _ => 0) (fun _ => 0) (driftVelocity Y) (driftPres
    FiniteCellEnergy (fun _ => 0) ∧ FiniteCellEnergy (driftVelocity Y) ∧
    (∃ z ∈ domain, driftVelocity Y z ≠ (0 : Vec))
```

The bounded drift

Credit: Builds on: the observation of Terence Tao (2011). Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (D), the repeating box, original wording, with no force and a fluid at rest. There are two distinct solutions from rest with no force: the still fluid, and one explicit drift whose speed and whose energy per unit cell stay below one. In symbols, for every $\nu > 0$:

$$\text{PrintedProblem}(\nu, 0, 0, 0, 0) \wedge \text{PrintedProblem}(\nu, 0, 0, \text{driftVelocity}\, Y_b, \text{driftPressu}$$

where Y_b is the book's bounded path.

> Lean code, for checking it yourself (the Lean names are internal labels, not terms of the problem). Checked by Lean, nanoda and lean4lean.

```
-- file: lean/JustExploring/BoundedSpeed.lean, line 59
theorem bounded_energy_nonuniqueness_in_printed_problem (v : ℝ) (hv : 0 < v
    PrintedProblem v (fun _ => 0) (fun _ => 0) (fun _ => 0) (fun _ => 0) ∧
    PrintedProblem v (fun _ => 0) (fun _ => 0)
      (driftVelocity boundedPath) (driftPressure boundedPath) ∧
    FiniteCellEnergy (driftVelocity boundedPath) ∧
    (∀ z ∈ domain, squaredSpeed (driftVelocity boundedPath z) < 1) ∧
    (∀ t ∈ times, (∫ x in cube, squaredSpeed (driftVelocity boundedPath (x
    (∃ z ∈ domain, driftVelocity boundedPath z ≠ (0 : Vec))
```

The slide-through escape

Credit: Builds on: Terence Tao (2011), the slide (Proposition 1.7; for all time, sketched in Remark 6.2 with the details omitted). Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (written).

Clay's question: (D), the repeating box, with a force, under the original 2000 wording (the velocity repeats; the pressure need not). Starting from rest, for every viscosity and every allowed force, a smooth flow slides through the box forever with a tilting pressure. So a forced breakdown from rest does not meet the printed (D); that includes OpenAI's construction. In symbols:

$$\forall \nu > 0,\ \forall f \text{ with } \text{ForceAdmissible}(f),\ \exists\, u, w, p, \pi, Y \text{ with } \text{EscapeConclusion}(\nu, f, \iota$$

The allowed forces are smooth, periodic and decaying in time as in (8), (9). The conclusion of the escape includes the printed solution class, a velocity formula and a pressure formula whose last term is the tilt:

$$u(z) = w\Big(z_1 - Y(z_2), z_2\Big) + Y'(z_2), \qquad p(z) = \pi\Big(z_1 - Y(z_2), z_2\Big) + \Big\langle \bar{f}(z_2) - Y$$

where $z = (z_1, z_2)$ with space part z_1 and time z_2, and $\bar{f}$ is the force's average.

> Lean code, for checking it yourself (the Lean names are internal labels, not terms of the problem). Checked by Lean, nanoda and lean4lean.

```
-- file: lean/JustCheckin/SlabStatements.lean, line 1146
theorem escape_from_rest : Statement.CorrectedTheorem3Statement
```

```
-- file: lean/JustCheckin/Statement.lean, line 148
def CorrectedTheorem3Statement : Prop :=
  ∀ v : ℝ, 0 < v → ∀ f : Velocity, ForceAdmissible f →
    ∃ (u w : Velocity) (p π : Pressure) (Y : ℝ → Space), EscapeConclusion
```

The conclusion of the escape includes the printed solution class, a velocity formula and a pressure formula whose last term is the tilt. The allowed forces are smooth, periodic and decaying in time as in (8), (9). The formulas, quoted from the Lean sources:

```
-- file: lean/JustCheckin/Statement.lean, EscapeConclusion at line 124, For
-- velocity: u z = w (z.1 - Y z.2, z.2) + speed Y z.2
-- pressure: p z = π (z.1 - Y z.2, z.2) + inner ℝ (meanForce f z.2 - accel
```

The escape-pressure criterion

Credit: Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (D) under the original wording, and what changes when the pressure must repeat. The pressure of the sliding escape repeats exactly when

the slide's acceleration equals the force's average at every time. In symbols, if $\mathrm{EscapeConclusion}(v, f, u, w, p, \pi, Y)$ holds, then

$$\mathrm{Periodic}(p) \iff \forall t \in \mathrm{times},\ Y''(t) = \bar{f}(t).$$

> Lean code, for checking it yourself (the Lean names are internal labels, not terms of the problem). Checked by Lean, nanoda and lean4lean.

```
-- file: lean/PressureEscapeRefutation.lean, line 9
theorem escape_pressure_periodic_iff (v : ℝ) (f u w : Velocity) (p π : Pres
    (Y : ℝ → Space) (h : Statement.EscapeConclusion v f u w p π Y) :
    Statement.Periodic p ↔ ∀ t ∈ times, Statement.acceleration Y t = State
```

The failure of the escape under repeating pressure

Credit: Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (D) under the erratum (the pressure must repeat). Once the pressure must repeat, the statement "every allowed force has such an escape" is false:

$$\neg\Big(\forall v > 0,\ \forall f,\ \mathrm{ForceAdmissible}(f) \to \exists\, u, w, p, \pi, Y,\ \mathrm{EscapeConclusion}(v, f, u, w$$

> Lean code, for checking it yourself (the Lean names are internal labels, not terms of the problem). Checked by Lean, nanoda and lean4lean.

```
-- file: lean/BoundaryMap.lean, line 28
theorem periodic_escape_refuted :
    ¬ (∀ v : ℝ, 0 < v → ∀ f : Velocity, Statement.ForceAdmissible f →
      ∃ (u w : Velocity) (p π : Pressure) (Y : ℝ → Space),
        Statement.EscapeConclusion v f u w p π Y ∧ Statement.Periodic p)
```

The forced breakdown with repeating pressure

Credit: Builds on: OpenAI's construction (2026); the mathematics of the construction is OpenAI's. Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): a check of another's result (verdict: proved (Lean)).

Clay's question: (D), the repeating box, with a force, under the erratum as intended (the velocity and the pressure both repeat). For every viscosity there is an allowed force for which no smooth flow from rest lasts forever with both velocity and pressure repeating:

$$\forall \nu > 0,\ \exists f,\ \mathrm{PrintedForce}(f) \wedge \neg \Big(\exists u, p,\ \mathrm{PrintedSolution}(\nu, f, u, p) \wedge \mathrm{Periodic}(p$$

Here $\mathrm{PrintedForce}(f)$ means: f is smooth; f is periodic; and for every $\alpha \in \{0, 1, \ \ldots\}^3$ (a multi-index), every m and every K there is $C > 0$ with $\|\partial_x^\alpha \partial_t^m f(z)\| \le C\,(1 + |z_2|)^{-K}$ for all z in the domain.

> Lean code, for checking it yourself (the Lean names are internal labels, not terms of the problem). Checked by Lean, nanoda and lean4lean.

```
-- file: lean/FinalTheorem.lean, line 8
theorem forced_periodic_from_rest : ForcedPeriodicStatement
```

```
-- file: lean/FinalSpec.lean, line 9
def ForcedPeriodicStatement : Prop :=
  ∀ v : ℝ, 0 < v → ∃ f : Velocity, PrintedForce f ∧
    ¬ (∃ u : Velocity, ∃ p : Pressure,
      Statement.PrintedSolution v f u p ∧ Statement.Periodic p)
```

```
-- file: lean/ForcePrinted.lean, line 11
structure PrintedForce (f : Velocity) : Prop where
  smooth : Smooth f
  periodic : Statement.Periodic f
  decay : ∀ (α : Fin 3 → ℕ) (m K : ℕ), ∃ C : ℝ, 0 < C ∧
    ∀ z ∈ domain, ‖Statement.mixedPartial α m f z‖ ≤ C * (1 + |z.2|) ^ (-(K
```

The two force classes are the same

Credit: Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (D), conditions (8) and (9) on the force. The force class used by the forced breakdown with repeating pressure is exactly the force class

used by the slide-through escape: for every f,
$\text{PrintedForce}(f) \iff \text{ForceAdmissible}(f)$.

> Lean code, for checking it yourself (the Lean names are internal labels, not terms of the problem). Checked by Lean (in the default build target), nanoda and lean4lean.

```
-- file: lean/ForcePrinted.lean, line 17
theorem printed_force_iff_admissible (f : Velocity) :
    PrintedForce f ↔ Statement.ForceAdmissible f
```

With no force, (B) under the printed pressure and under periodic pressure are the same question

Credit: Builds on: Terence Tao (2011, p. 6 and Lemma 4.1). Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (B), the repeating box, no force. For every starting flow u_0 and every ν, with no force, a smooth solution with periodic velocity exists exactly when one with periodic velocity and periodic pressure exists:

$$\Big(\exists u,p,\ \text{PrintedSolution}(\nu,u_0,0,u,p)\Big) \iff \Big(\exists u,p,\ \text{CorrectedSolution}(\nu,u_0,0,u$$

> Lean code, for checking it yourself (the Lean names are internal labels, not terms of the problem). Checked by Lean (in the default build target), nanoda and lean4lean.

```
-- file: lean/InheritedBridge.lean, line 49
theorem inherited_existence_iff (nu : ℝ) (u0 : Space → Space) :
    (∃ u p, InheritedPressure.PrintedSolution nu u0 (fun _ => 0) u p) ↔
      (∃ u p, InheritedPressure.CorrectedSolution nu u0 (fun _ => 0) u p)
```

The same breakdown over wider classes of forces and starting flows

Clay's question. (D), the repeating box, with a force, under the erratum.

Answer. The breakdown also holds for every smooth repeating force (this class has no decay condition, so it is a corollary rather than a statement

shaped like Clay's (D)), for the repeating force class of the Formal Conjectures project (with a zero start), and for every repeating starting flow together with that force class. The Formal Conjectures version requires the pressure to repeat, "following the errata appended to the Clay problem statement". The version over every repeating starting flow keeps a universal quantifier over the starting flow inside the negation, so it does not by itself name the starting flow; the forced breakdown with repeating pressure names it (rest).

> Lean code, for checking it yourself (the Lean names are internal labels, not terms of the problem). Checked by Lean, nanoda and lean4lean. The comparator definitions it uses are named in the code comment.

```
-- file: lean/BoundaryMap.lean, lines 49, 61 and 71
-- (the comparator definitions used below: lean/NavierStokes/ComparatorDefi
theorem periodic_smooth_existence_refuted (v : ℝ) (hv : 0 < v) :
    ¬ (∀ f : Velocity, Smooth f → Statement.Periodic f →
      ∃ u : Velocity, ∃ p : Pressure,
        Statement.PrintedSolution v f u p ∧ Statement.Periodic p)

theorem periodic_decay_existence_refuted (v : ℝ) (hv : 0 < v) :
    ¬ (∀ g : Space → ℝ → Space, NavierStokes.Comparator.ForceConditionPeri
      ∃ v p, NavierStokes.Comparator.NavierStokesExistenceAndSmoothnessPer
        v (fun _ => 0) g v p)

theorem openai_D_universal_existence_refuted (v : ℝ) (hv : 0 < v) :
    ¬ (∀ u₀ : Space → Space, NavierStokes.Comparator.InitialVelocityCondit
      ∀ g : Space → ℝ → Space, NavierStokes.Comparator.ForceConditionPerio
      ∃ v p, NavierStokes.Comparator.NavierStokesExistenceAndSmoothnessPer
```

The whole-space breakdown

Credit: Builds on: OpenAI's whole-space construction (2026); the mathematics of the construction is OpenAI's. Tools: Lean 4 and Mathlib; the nanoda kernel checker. This result (Sonny): a check of another's result (verdict: proved (Lean)).

Clay's question: (C), all of space, with a force: show that some flow breaks down. For every viscosity there is a decaying starting flow and a decaying force on all of space for which no smooth solution exists for all time, in the

quantifier shape of the Formal Conjectures comparator: for every $\nu > 0$ there exist $u_0 : R^3 \to R^3$ and $f : R^3 \times R \to R^3$ with InitialVelocityConditionDecay(u_0), ForceConditionDecay(f) and $\neg \Big(\exists v, p,\ \text{NavierStokesExistenceAndSmoothnessRn}(\nu, u_0, f, v, p) \Big)$.

> Lean code, for checking it yourself (the Lean names are internal labels, not terms of the problem). Checked by Lean (Lean 4.34.1, Mathlib d13f23b7; the printed axioms are propext, Classical.choice and Quot.sound) and nanoda, which accepted an export of 94,908 declarations. It is built in a separate target from OpenAI's whole-space part of the same package, with one overlay file used in the checked build (named in the code comment). The clause-by-clause comparison with Clay's printed (C) is in the comparison note.

```
-- file: lean/NavierStokes/ComparatorR3Theorem.lean, line 38
-- (overlay used in the checked build: lean/overlays/NavierStokes/Comparato
theorem navier_stokes_breakdown_R3 (ν : ℝ) (hν : ν > 0) :
    ∃ (u₀ : EuclideanSpace ℝ (Fin 3) → EuclideanSpace ℝ (Fin 3))
      (f : EuclideanSpace ℝ (Fin 3) → ℝ → EuclideanSpace ℝ (Fin 3)),
      Comparator.InitialVelocityConditionDecay u₀ ∧ Comparator.ForceConditi
      ¬ (∃ v p, Comparator.NavierStokesExistenceAndSmoothnessRn ν u₀ f v p
```

Collections of Lean helper statements (first round)

These collections are Sonny's (2026). The literal-erratum bridge joins those four lemmas to OpenAI's forced breakdown with repeating pressure. The statements are in the Lean files listed in the code block below; each was compiled when it was written, with only the three standard axioms. The Lean names are internal labels, not terms of the problem.

- A collection of 62 theorems and another of 138 theorems: parts of the lemmas of chapter 8, section 5. One file in the second collection prints the axioms.
- The literal-erratum statements of chapter 6 (four lemmas) and the bridge joining them to the forced breakdown with repeating pressure.

- A group of 45 declarations: helper statements for the closure of the weak-solution and degenerate-identity readings. The scope of each is in the formal scope note.
- A group of 58 declarations from the force-mean group: the construction's force has exactly zero cell mean at every time; the exclusion of zero-mean-velocity competitors; the exclusion for a force fixed across viscosities; the equivalence of pressure-free flows with incompressible Burgers flows; and the energy-mean algebra. One file prints their axioms.
- A group of 7 theorems: the period-defect system for periodic data and the face algebra used for the one-cell identity question.

```
lean/NSLemmas/
lean/NSContinuation/            (AxiomRun.lean prints the axioms)
lean/NSClose/Literal.lean
lean/NSClose/Bridge.lean
lean/CloseBridge.lean  lean/CloseReadings.lean  lean/InitialWitness.lean
lean/TraceObstruction.lean  lean/WitnessJets.lean
lean/FullMean.lean  lean/PrintedClassScope.lean  lean/UniformScope.lean
lean/PressureFreeScope.lean  lean/EnergyMean.lean
lean/AxiomAudit.lean
lean/LocalResearch.lean
```

Collections of Lean helper statements (second round)

These collections are Sonny's (2026). Several are about OpenAI's construction or use it (the closures of (C), the growth theorem for OpenAI's force, Tao's Conjecture 1.8 and the curl of OpenAI's force); the construction itself is OpenAI's. Every printed axiom list is only propext, Classical.choice and Quot.sound; nanoda accepted the exports named. The build targets and commands are in the check guide. The Lean names are internal labels, not terms of the problem.

- Clay's (A), (B) and (D) under the 2000 wording, stated directly against the comparator definitions, with the extra-identity reduction for smooth solutions (an export of 93,579 declarations); and the two growth theorems of chapter 6 (91,888).

- The whole-space closures of (C): the zero-extended identity, the ordinary printed (C) variants, the logic of the readings with one force for every viscosity, and the weak zero-mean combination (95,425).
- Weak solutions: the trace weak class and Clay's (A) to (D) with weak solutions (50,100).
- One force for every viscosity: the scaling, the classes and the reduction to small-data existence at viscosity one (53,132).
- The scaling lemma: no nonzero force of class (5) is unchanged by a nontrivial rescaling of the equations (33,099).

- Tao's Conjecture 1.8: Sonny's Lean proof (2026) that OpenAI's construction refutes it as Tao wrote it, under each of three readings of his normalised pressure and for a solution of any spatial period (chapter 4); it uses OpenAI's periodic Corollary 10.6 (91,179 for period 1; 95,687 for any period). The two written steps it relies on are in the Conjecture 1.8 written steps.
- The written-rows part of chapter 8, section 5: the smooth-solutions identity, the curl of OpenAI's force for every viscosity with no premise, the start-level readings of (B) and (D) with starts of least period one and with a zero-mean start, and the pressure-on-all-of-space lemma for solutions smooth up to $t = 0$ and for solutions smooth for $t > 0$ with the pressure repeating at every positive time (94,345).

That makes seven collections in the second round. The one-push files, the scaling lemma, Tao's Conjecture 1.8 and the written-rows part were built from the shipped build file, with the other targets.

```
lean/OWDefs.lean  lean/OWA.lean  lean/OWShear.lean  lean/OWA2.lean  lean/OW
lean/LinearGrowthCalc.lean  lean/LinearGrowthIdentity.lean  lean/LinearGrow
lean/EnergyAE.lean  lean/SettleCommonForce.lean  lean/OrdinaryVariants.lean
lean/ZeroExtendedCore.lean  lean/ZeroExtendedDeriv.lean  lean/ZeroExtended
lean/WCCore.lean  lean/WCProd.lean  lean/WCClass.lean  lean/WCWitness.lean
lean/CFScaling.lean  lean/CFClasses.lean  lean/CFReduction.lean  lean/CFSta
lean/ScalingInvariantForce.lean
lean/TaoConjecture18.lean  lean/TaoConjecture18Period.lean
lean/WRPressure.lean  lean/WRStart.lean  lean/WRWeak.lean  lean/WRBridge.le
```

Where to go next

- Which reading does each statement settle? See Every reading.
- How do you rebuild the Lean files and print their axioms? See the check guide.

13 Theorems in this book

Theorems in this book

Every theorem and lemma in this book, with its exact statement, status and credit. Maths is written in LaTeX notation, between dollar signs.

The entries follow the book's own order: the twelve Lean-checked statements of the exact statements page; the Conjecture 1.8 result of chapter 4; the results of chapter 6; the fifteen lemmas of chapter 8, section 5; the cited classics; the closure results of the section "Written results behind section 4" of chapter 8 and of the proofs folder; then the open questions. Theorems 63 and 64 are results of chapter 6; they come after Theorem 62 so that the other numbers stay as cited. The Lean names are internal labels, not terms of the problem. Chapter 8 section 5 gives complete written proofs for its lemmas; here each lemma is stated and the proof is linked.

Clay's clauses, as the entries use them

Quoted from the exact statements page, Part 1 (mathematical symbols transcribed). Clay's problem asks about $\nu > 0$ and $n = 3$.

Equation (1): $\partial_t u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t)$ for $x \in R^n$, $t \geq 0$.

Equation (2): $\mathrm{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0$. Initial condition (3): $u(x,0) = u^\circ(x)$ for $x \in R^n$, with u° a given C^∞ divergence-free vector field, f_i the components of a given externally applied force and ν a positive coefficient, the viscosity.

Whole-space conditions: (4) $|\partial_x^\alpha u^\circ(x)| \leq C_{\alpha K}(1+|x|)^{-K}$ on R^n, for any α and K; (5) $|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}$ on $R^n \times [0,\infty)$, for any α, m, K; (6)

$p, u \in C^\infty(R^n \times [0,\infty))$; (7) $\int_{R^n} |u(x,t)|^2\, dx < C$ for all $t \geq 0$ (bounded energy).

Periodic conditions: (8) $u^\circ(x + e_j) = u^\circ(x)$ and $f(x + e_j, t) = f(x,t)$ for $1 \leq j \leq n$ (e_j the \$j\$th unit vector); (9) $|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1 + |t|)^{-K}$ on $R^3 \times [0,\infty)$, for any α, m, K; (10) $u(x,t) = u(x + e_j, t)$ on $R^3 \times [0,\infty)$ for $1 \leq j \leq n$; (11) $p, u \in C^\infty(R^n \times [0,\infty))$.

Weak form: for any smooth vector field θ compactly supported in $R^3 \times (0,\infty)$, (12)
$\iint u \cdot \partial_t \theta\, dx\, dt - \sum_{ij} \iint u_i u_j\, \partial_{x_j} \theta_i\, dx\, dt = \nu \iint u \cdot \Delta\theta\, dx\, dt + \iint f \cdot \theta\, dx\, dt - \iint$
; and for smooth φ compactly supported in $R^3 \times (0,\infty)$, (13)
$\iint u \cdot \nabla_x \varphi\, dx\, dt = 0$. The erratum: "The further condition $p(x + e_j, t) = p(x,t)$ should be made explicit in Eqn (8)", and "Eqn (10) should read" $-\iint u \cdot \partial_t \theta - \sum_{ij} \iint u_i u_j\, \partial_{x_j} \theta_i = \nu \iint u \cdot \Delta\theta + \iint f \cdot \theta + \iint p\, \mathrm{div}\theta$.

The four alternatives. (A), $\nu > 0$, $n = 3$, u° any smooth divergence-free field satisfying (4), $f \equiv 0$: there exist smooth p, u_i on $R^3 \times [0,\infty)$ satisfying (1), (2), (3), (6), (7). (B), $\nu > 0$, $n = 3$, u° any smooth divergence-free field satisfying (8), $f \equiv 0$: there exist smooth p, u_i on $R^3 \times [0,\infty)$ satisfying (1), (2), (3), (10), (11). (C), $\nu > 0$, $n = 3$: there exist a smooth divergence-free u° on R^3 and a smooth f on $R^3 \times [0,\infty)$ satisfying (4), (5), for which there exist no solutions (p,u) of (1), (2), (3), (6), (7) on $R^3 \times [0,\infty)$. (D), $\nu > 0$, $n = 3$: there exist a smooth divergence-free u° on R^3 and a smooth f on $R^3 \times [0,\infty)$ satisfying (8), (9), for which there exist no solutions (p,u) of (1), (2), (3), (10), (11) on $R^3 \times [0,\infty)$.

The Lean definitions behind the twelve opening entries, quoted word for word from the exact statements page. "Periodic F" means that F repeats under a shift by one in each space direction, on $R^3 \times [0,\infty)$. "Smooth" means C^∞ up to $t = 0$ on the closed half-space.

```
-- file: lean/JustCheckin/Statement.lean, line 58
structure PrintedSolution (ν : ℝ) (f u : Velocity) (p : Pressure) : Prop wh
  velocity_smooth : Smooth u
  pressure_smooth : Smooth p
  momentum : ∀ z ∈ domain, residual ν f u p z = 0
  incompressible : ∀ z ∈ domain, divergence u z = 0
  initial_rest : ∀ x : Space, u (x, 0) = 0
  velocity_periodic : Periodic u

-- file: lean/JustCheckin/Geometry.lean, lines 35 and 42
def domain : Set SpaceTime := univ ×ˢ Ici (0 : ℝ)
def Smooth {E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E]
    (F : SpaceTime → E) : Prop := ContDiffOn ℝ ∞ F domain

-- file: lean/JustCheckin/Statement.lean, lines 52 and 55
def divergence (u : Velocity) (z : SpaceTime) : ℝ :=
  Σ j : Fin 3, «partial» j u z j
def residual (ν : ℝ) (f u : Velocity) (p : Pressure) (z : SpaceTime) : Spa
  timePartial u z + derivative u z (u z, 0) - ν • laplacian u z + gradient
```

In the entries below, residual is $\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p - f$, which is equation (1) with every term moved to the left.

Theorem 1: the conditional zero-force drift (Tao, 2011; built on by Sonny, 2026)

Credit: Builds on: the observation of Terence Tao (2011). Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (D), the repeating box, with a force, under the original wording. This result is about the printed wording of (10) and (11), which does not make the pressure repeat. With no force and a fluid at rest, the printed wording allows both the still fluid and a drifting flow, provided one has a smooth path that starts with zero position and zero velocity and is not zero later. The drifting flow is conditional on that path. In symbols: for every $\nu > 0$ and every smooth path Y with $Y(0) = 0$, speed $Y'(0) = 0$ and $Y(t) \neq 0$ for some t in the time set,

$$\text{PrintedProblem}(\nu, 0, 0, 0, 0) \;\wedge\; \text{PrintedProblem}(\nu, 0, 0, \text{driftVelocity}\, Y, \text{driftPress}$$

Status: proved in Lean (checked by Lean, nanoda and lean4lean). The drifting flow is conditional on the path Y.

```
-- name: ClayWording.zero_data_nonuniqueness_in_printed_problem
-- file: lean/JustExploring/Data.lean, line 55
theorem zero_data_nonuniqueness_in_printed_problem (v : ℝ) (hv : 0 < v)
    {Y : ℝ → Vec} (hY : SmoothPath Y) (hY0 : Y 0 = 0) (hv0 : speed Y 0 = 0)
    (hnonzero : ∃ t ∈ times, Y t ≠ 0) :
    PrintedProblem v (fun _ => 0) (fun _ => 0) (fun _ => 0) (fun _ => 0) ∧
    PrintedProblem v (fun _ => 0) (fun _ => 0) (driftVelocity Y) (driftPres
    FiniteCellEnergy (fun _ => 0) ∧ FiniteCellEnergy (driftVelocity Y) ∧
    (∃ z ∈ domain, driftVelocity Y z ≠ (0 : Vec))
```

Proof: chapter 4 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-04-tao-2011); statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements), "The conditional zero-force drift"; Lean theorem named in the code block above.

Theorem 2: the bounded drift (Tao, 2011; built on by Sonny, 2026)

Credit: Builds on: the observation of Terence Tao (2011). Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (D), the repeating box, original wording, with no force and a fluid at rest. There are two distinct solutions from rest with no force: the still fluid, and one explicit drift whose speed and whose energy per unit cell stay below one. In symbols, for every $\nu > 0$:

$$\text{PrintedProblem}(\nu, 0, 0, 0, 0) \wedge \text{PrintedProblem}(\nu, 0, 0, \text{driftVelocity}\, Y_b, \text{driftPressu}$$

where Y_b is the book's bounded path.

Status: proved in Lean (checked by Lean, nanoda and lean4lean).

```
-- name: ClayWording.bounded_energy_nonuniqueness_in_printed_problem
-- file: lean/JustExploring/BoundedSpeed.lean, line 59
theorem bounded_energy_nonuniqueness_in_printed_problem (v : ℝ) (hv : 0 < v
    PrintedProblem v (fun _ => 0) (fun _ => 0) (fun _ => 0) (fun _ => 0) ∧
    PrintedProblem v (fun _ => 0) (fun _ => 0)
      (driftVelocity boundedPath) (driftPressure boundedPath) ∧
    FiniteCellEnergy (driftVelocity boundedPath) ∧
    (∀ z ∈ domain, squaredSpeed (driftVelocity boundedPath z) < 1) ∧
    (∀ t ∈ times, (∫ x in cube, squaredSpeed (driftVelocity boundedPath (x
    (∃ z ∈ domain, driftVelocity boundedPath z ≠ (0 : Vec))
```

Proof: chapter 4 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-04-tao-2011); statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements), "The bounded drift"; Lean theorem named in the code block above.

Theorem 3: the slide-through escape

Credit: Builds on: Terence Tao (2011), the slide (Proposition 1.7; for all time, sketched in Remark 6.2 with the details omitted). Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (written).

Clay's question: (D), the repeating box, with a force, under the original 2000 wording (the velocity repeats; the pressure need not). Starting from rest, for every viscosity and every allowed force, a smooth flow slides through the box forever with a tilting pressure. So a forced breakdown from rest does not meet the printed (D); that includes OpenAI's construction. In symbols:

$$\forall \nu > 0,\ \forall f \text{ with } \mathrm{ForceAdmissible}(f),\ \exists\, u, w, p, \pi, Y \text{ with } \mathrm{EscapeConclusion}(\nu, f, u$$

The allowed forces are smooth, periodic and decaying in time as in (8), (9). The conclusion of the escape includes the printed solution class, a velocity formula and a pressure formula whose last term is the tilt:

$$u(z) = w\Big(z_1 - Y(z_2), z_2\Big) + Y'(z_2), \qquad p(z) = \pi\Big(z_1 - Y(z_2), z_2\Big) + \Big\langle \bar{f}(z_2) - Y$$

where $z = (z_1, z_2)$ with space part z_1 and time z_2, and $\bar{f}$ is the force's average.

Status: written; partly Lean: proved in Lean (checked by Lean, nanoda and lean4lean), except one step: its H^3 bound is stated in Lean, but without the clause that its series converges; that step is checked in writing.

```
-- name: JustCheckin.A3.escape_from_rest
-- file: lean/JustCheckin/SlabStatements.lean, line 1146
theorem escape_from_rest : Statement.CorrectedTheorem3Statement

-- file: lean/JustCheckin/Statement.lean, line 148
def CorrectedTheorem3Statement : Prop :=
  ∀ ν : ℝ, 0 < ν → ∀ f : Velocity, ForceAdmissible f →
    ∃ (u w : Velocity) (p π : Pressure) (Y : ℝ → Space), EscapeConclusion

-- file: lean/JustCheckin/Statement.lean, EscapeConclusion at line 124, For
-- velocity: u z = w (z.1 - Y z.2, z.2) + speed Y z.2
-- pressure: p z = π (z.1 - Y z.2, z.2) + inner ℝ (meanForce f z.2 - accel
```

Proof: chapter 4 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-04-tao-2011); statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements), "The slide-through escape"; Lean theorem named in the code block above.

Full theorem and proof: proofs/FULL-FROM-REST-D.md

Theorem 4: the escape-pressure criterion

Credit: Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (D) under the original wording, and what changes when the pressure must repeat. The pressure of the sliding escape repeats exactly when the slide's acceleration equals the force's average at every time. In symbols, if $\mathrm{EscapeConclusion}(\nu, f, u, w, p, \pi, Y)$ holds, then

$$\mathrm{Periodic}(p) \iff \forall t \in \mathrm{times},\ Y''(t) = \bar{f}(t).$$

Status: proved in Lean (checked by Lean, nanoda and lean4lean).

```
-- name: JustCheckin.A3.PressureBoundary.escape_pressure_periodic_iff
-- file: lean/PressureEscapeRefutation.lean, line 9
theorem escape_pressure_periodic_iff (v : ℝ) (f u w : Velocity) (p π : Pres
    (Y : ℝ → Space) (h : Statement.EscapeConclusion v f u w p π Y) :
    Statement.Periodic p ↔ ∀ t ∈ times, Statement.acceleration Y t = State
```

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open); statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements), "The escape-pressure criterion"; Lean theorem named in the code block above.

Theorem 5: the failure of the escape under repeating pressure

Credit: Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (D) under the erratum (the pressure must repeat). Once the pressure must repeat, the statement "every allowed force has such an escape" is false:

$$\neg\Big(\forall \nu > 0,\ \forall f,\ \mathrm{ForceAdmissible}(f) \to \exists\, u, w, p, \pi, Y,\ \mathrm{EscapeConclusion}(\nu, f, u, w$$

Status: proved in Lean (checked by Lean, nanoda and lean4lean).

```
-- name: JustCheckin.BoundaryMap.periodic_escape_refuted
-- file: lean/BoundaryMap.lean, line 28
theorem periodic_escape_refuted :
    ¬ (∀ v : ℝ, 0 < v → ∀ f : Velocity, Statement.ForceAdmissible f →
      ∃ (u w : Velocity) (p π : Pressure) (Y : ℝ → Space),
        Statement.EscapeConclusion v f u w p π Y ∧ Statement.Periodic p)
```

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open); statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements), "The failure of the escape under repeating pressure"; Lean theorem named in the code block above.

Theorem 6: the forced breakdown with repeating pressure (OpenAI, 2026)

Credit: Builds on: OpenAI's construction (2026); the mathematics of the construction is OpenAI's. Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): a check of another's result (verdict: proved (Lean)).

Clay's question: (D), the repeating box, with a force, under the erratum as intended (the velocity and the pressure both repeat). For every viscosity there is an allowed force for which no smooth flow from rest lasts forever with both velocity and pressure repeating:

$$\forall \nu > 0,\ \exists f,\ \mathrm{PrintedForce}(f) \wedge \neg \Big(\exists u, p,\ \mathrm{PrintedSolution}(\nu, f, u, p) \wedge \mathrm{Periodic}(p)$$

Here $\mathrm{PrintedForce}(f)$ means: f is smooth; f is periodic; and for every $\alpha \in \{0, 1, \ \ldots\}^3$ (a multi-index), every m and every K there is $C > 0$ with $\|\partial_x^\alpha \partial_t^m f(z)\| \le C\,(1 + |z_2|)^{-K}$ for all z in the domain.

Status: proved in Lean (checked by Lean, nanoda and lean4lean).

```
-- name: ForcedBreakdown.forced_periodic_from_rest
-- file: lean/FinalTheorem.lean, line 8
theorem forced_periodic_from_rest : ForcedPeriodicStatement

-- file: lean/FinalSpec.lean, line 9
def ForcedPeriodicStatement : Prop :=
  ∀ ν : ℝ, 0 < ν → ∃ f : Velocity, PrintedForce f ∧
    ¬ (∃ u : Velocity, ∃ p : Pressure,
      Statement.PrintedSolution ν f u p ∧ Statement.Periodic p)

-- file: lean/ForcePrinted.lean, line 11
structure PrintedForce (f : Velocity) : Prop where
  smooth : Smooth f
  periodic : Statement.Periodic f
  decay : ∀ (α : Fin 3 → ℕ) (m K : ℕ), ∃ C : ℝ, 0 < C ∧
    ∀ z ∈ domain, ‖Statement.mixedPartial α m f z‖ ≤ C * (1 + |z.2|) ^ (-(K
```

Proof: chapter 3 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-03-openai-2026);

statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements), "The forced breakdown with repeating pressure"; Lean theorem named in the code block above.

Theorem 7: the two force classes are the same

Credit: Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (D), conditions (8) and (9) on the force. The force class used by the forced breakdown with repeating pressure is exactly the force class used by the slide-through escape: for every f, $\mathrm{PrintedForce}(f) \iff \mathrm{ForceAdmissible}(f)$.

Status: proved in Lean (checked by Lean, in the default build target, nanoda and lean4lean).

```
-- name: ForcedBreakdown.printed_force_iff_admissible
-- file: lean/ForcePrinted.lean, line 17
theorem printed_force_iff_admissible (f : Velocity) :
    PrintedForce f ↔ Statement.ForceAdmissible f
```

Proof: statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements), "The two force classes are the same"; Lean theorem named in the code block above.

Theorem 8: with no force, (B) under the printed pressure and under periodic pressure are the same question (Tao, 2011; Sonny, 2026)

Credit: Builds on: Terence Tao (2011, p. 6 and Lemma 4.1). Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (Lean).

Clay's question: (B), the repeating box, no force. For every starting flow u_0 and every ν, with no force, a smooth solution with periodic velocity exists exactly when one with periodic velocity and periodic pressure exists:

$$\left(\exists u, p,\ \text{PrintedSolution}(\nu, u_0, 0, u, p)\right) \iff \left(\exists u, p,\ \text{CorrectedSolution}(\nu, u_0, 0, u\right.$$

Status: proved in Lean (checked by Lean, in the default build target, nanoda and lean4lean).

```
-- name: JustExploringInherited.inherited_existence_iff
-- file: lean/InheritedBridge.lean, line 49
theorem inherited_existence_iff (nu : ℝ) (u0 : Space → Space) :
    (∃ u p, InheritedPressure.PrintedSolution nu u0 (fun _ => 0) u p) ↔
      (∃ u p, InheritedPressure.CorrectedSolution nu u0 (fun _ => 0) u p)
```

Proof: chapter 5 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-05-other-answers-under-the-erratum); statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements), "With no force, (B) under the printed pressure and under periodic pressure are the same question"; Lean theorem named in the code block above.

Full theorem and proof: proofs/FULL-B-EQUIVALENCE.md

Theorem 9: the breakdown for every smooth repeating force (OpenAI's construction; stated and checked in Lean by Sonny)

Credit: Builds on: OpenAI's construction (2026); the mathematics of the construction is OpenAI's. Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): a formalisation of another's result (verdict: proved (Lean)).

Clay's question: (D), the repeating box, with a force, under the erratum. The breakdown also holds for every smooth repeating force (this class has no decay condition, so it is a corollary rather than a statement shaped like Clay's (D)). For $\nu > 0$:

$$\neg\left(\forall f,\ \text{Smooth}(f) \wedge \text{Periodic}(f) \to \exists u, p,\ \text{PrintedSolution}(\nu, f, u, p) \wedge \text{Periodic}(\right.$$

Status: proved in Lean (checked by Lean, nanoda and lean4lean).

```
-- name: JustCheckin.BoundaryMap.periodic_smooth_existence_refuted
-- file: lean/BoundaryMap.lean, line 49
-- (the comparator definitions used below: lean/NavierStokes/ComparatorDef
theorem periodic_smooth_existence_refuted (v : ℝ) (hv : 0 < v) :
    ¬ (∀ f : Velocity, Smooth f → Statement.Periodic f →
      ∃ u : Velocity, ∃ p : Pressure,
        Statement.PrintedSolution v f u p ∧ Statement.Periodic p)
```

Proof: chapter 3 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-03-openai-2026); statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements), "The same breakdown over wider classes of forces and starting flows"; Lean theorem named in the code block above.

Theorem 10: the breakdown in the Formal Conjectures force class (OpenAI's construction; stated and checked in Lean by Sonny)

Credit: Builds on: OpenAI's construction (2026); the mathematics of the construction is OpenAI's; the Formal Conjectures comparator. Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): a formalisation of another's result (verdict: proved (Lean)).

Clay's question: (D), the repeating box, with a force, under the erratum. The breakdown holds for the repeating force class of the Formal Conjectures project (with a zero start). The Formal Conjectures version requires the pressure to repeat, "following the errata appended to the Clay problem statement". For $\nu > 0$:

$$\neg\Big(\forall g,\ \mathrm{ForceConditionPeriodic}(g) \to \exists v, p,\ \mathrm{NavierStokesExistenceAndSmoothr$$

Status: proved in Lean (checked by Lean, nanoda and lean4lean). The comparator definitions it uses are named in the code comment.

```
-- name: JustCheckin.BoundaryMap.periodic_decay_existence_refuted
-- file: lean/BoundaryMap.lean, line 61
theorem periodic_decay_existence_refuted (ν : ℝ) (hν : 0 < ν) :
    ¬ (∀ g : Space → ℝ → Space, NavierStokes.Comparator.ForceConditionPeri
      ∃ v p, NavierStokes.Comparator.NavierStokesExistenceAndSmoothnessPer
        ν (fun _ => 0) g v p)
```

Proof: chapter 3 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-03-openai-2026); statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements); Lean theorem named in the code block above.

Theorem 11: the breakdown for every repeating starting flow (OpenAI's construction; stated and checked in Lean by Sonny)

Credit: Builds on: OpenAI's construction (2026); the mathematics of the construction is OpenAI's; the Formal Conjectures comparator. Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): a formalisation of another's result (verdict: proved (Lean)).

Clay's question: (D), the repeating box, with a force, under the erratum. The breakdown holds for every repeating starting flow together with that force class. The version over every repeating starting flow keeps a universal quantifier over the starting flow inside the negation, so it does not by itself name the starting flow; the forced breakdown with repeating pressure names it (rest). For $\nu > 0$:

$$\neg\Big(\forall u_0,\ \mathrm{InitialVelocityConditionPeriodic}(u_0) \to \forall g,\ \mathrm{ForceConditionPeriodic}(g$$

Status: proved in Lean (checked by Lean, nanoda and lean4lean).

```
-- name: JustCheckin.BoundaryMap.openai_D_universal_existence_refuted
-- file: lean/BoundaryMap.lean, line 71
theorem openai_D_universal_existence_refuted (v : ℝ) (hv : 0 < v) :
    ¬ (∀ u₀ : Space → Space, NavierStokes.Comparator.InitialVelocityCondit
      ∀ g : Space → ℝ → Space, NavierStokes.Comparator.ForceConditionPerio
      ∃ v p, NavierStokes.Comparator.NavierStokesExistenceAndSmoothnessPer
```

Proof: chapter 3 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-03-openai-2026); statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements); Lean theorem named in the code block above.

Theorem 12: the whole-space breakdown (built from OpenAI's whole-space construction, 2026)

Credit: Builds on: OpenAI's whole-space construction (2026); the mathematics of the construction is OpenAI's. Tools: Lean 4 and Mathlib; the nanoda kernel checker. This result (Sonny): a check of another's result (verdict: proved (Lean)).

Clay's question: (C), all of space, with a force: show that some flow breaks down. For every viscosity there is a decaying starting flow and a decaying force on all of space for which no smooth solution exists for all time, in the quantifier shape of the Formal Conjectures comparator: for every $\nu > 0$ there exist $u_0 : R^3 \to R^3$ and $f : R^3 \times R \to R^3$ with $\mathrm{InitialVelocityConditionDecay}(u_0)$, $\mathrm{ForceConditionDecay}(f)$ and $\neg\left(\exists v, p,\ \mathrm{NavierStokesExistenceAndSmoothnessRn}(\nu, u_0, f, v, p)\right)$.

Status: proved in Lean (checked by Lean 4.34.1 with Mathlib d13f23b7; the printed axioms are propext, Classical.choice and Quot.sound) and nanoda, which accepted an export of 94,908 declarations. It is built in a separate target from OpenAI's whole-space part of the same package, with one overlay file used in the checked build.

```
-- name: NavierStokes.ComparatorBridge.navier_stokes_breakdown_R3
-- file: lean/NavierStokes/ComparatorR3Theorem.lean, line 38
-- (overlay used in the checked build: lean/overlays/NavierStokes/Comparato
theorem navier_stokes_breakdown_R3 (ν : ℝ) (hν : ν > 0) :
    ∃ (u₀ : EuclideanSpace ℝ (Fin 3) → EuclideanSpace ℝ (Fin 3))
      (f : EuclideanSpace ℝ (Fin 3) → ℝ → EuclideanSpace ℝ (Fin 3)),
      Comparator.InitialVelocityConditionDecay u₀ ∧ Comparator.ForceConditi
      ¬ (∃ v p, Comparator.NavierStokesExistenceAndSmoothnessRn ν u₀ f v p
```

Proof: chapter 3 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-03-openai-2026); proofs/C-CLAUSES.md; proofs/C-FORMAL-CHECKS.md; Lean theorem named in the code block above.

Theorem 13: Corollary 10.6, the periodic breakdown (OpenAI, 2026, cited)

Credit: OpenAI (2026), p. 125 of *Finite time blowup for Navier-Stokes*. The Lean version is OpenAI's own theorem, rebuilt and checked in this book.

For every $\nu > 0$, there is a smooth force on $T^3 \times [0, \infty)$, compactly supported in time, for which the solution (U, P_{per}) of the periodic Navier-Stokes equations with viscosity ν and zero initial velocity is smooth on $[0, 1)$ and satisfies

$$\limsup_{t \uparrow 1} \|U(t)\|_{L^\infty(T^3)} = \infty.$$

Its proof ends with a uniqueness step: for two smooth periodic solutions with the same force and datum, "Periodic integration removes both the transport and pressure terms." Removing the pressure term this way uses periodicity of the pressure.

Status: cited (OpenAI's theorem; in this book it is the Lean theorem of the forced breakdown with repeating pressure and its supporting package, checked by Lean, nanoda and lean4lean).

Proof: chapter 3 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-03-openai-2026), section "For specialists";

statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements).

Proposition 14: Tao's Proposition 1.7, elimination of the forcing term (Tao, 2011, cited)

Credit: Terence Tao (2011), arXiv:1108.1165, Proposition 1.7 [v1: 1.12]; the version for all time is sketched in his Remark 6.2 with the details omitted. Cited, not rechecked in Lean.

The forced periodic conjecture is equivalent to the unforced one. On finite time intervals this is the proposition; Tao: "Proposition 1.7 exploits the technical loophole of non-periodic pressure." It rests on the Galilean symmetry of the equations. For other starts than rest, on every finite time interval, (D) holds exactly when (B) fails, by Tao's Proposition 1.7 (2011); for all time this holds only if the version his Remark 6.2 sketches, with the details omitted, holds; cited, not rechecked in Lean.

Status: cited (not rechecked in Lean).

Proof: chapter 4 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-04-tao-2011), section "For specialists".

Lemma 15: the pressure of a smooth periodic solution (Tao, 2011, Lemma 4.1(ii), cited)

Credit: Terence Tao (2011), Lemma 4.1(ii). Cited.

The pressure of a smooth periodic solution is the normalised pressure plus an affine term, $p = p_{\text{norm}} + x \cdot a(t) + C(t)$, where the normalised pressure is $p = -\Delta^{-1}\partial_i\partial_j(u_i u_j) + \Delta^{-1}\nabla \cdot f$.

Status: cited.

Proof: chapter 4 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-04-tao-2011), section "For specialists".

Theorem 16: Tao's Conjecture 1.8 is false as Tao wrote it

Credit: Sonny, applying OpenAI's construction (Corollary 10.6, 2026) to Terence Tao's Conjecture 1.8 (2011, with footnote 4): the deduction for every period and every reading, its Lean check, and the two short written steps that connect the Lean definitions to Tao's words.

Tao's Conjecture 1.8 (arXiv:1108.1165v4, p. 5 [v1: 1.13]), global regularity for periodic data with normalised pressure: "Let (u_0, f, T) be a smooth periodic set of data. Then there exists a smooth periodic solution (u, p, u_0, f, T) with the indicated data and with normalised pressure." The answer is no. Start from rest with OpenAI's push at viscosity 1, which is smooth and repeats with period 1. For every end time $T \geq 1$ there is no smooth periodic solution on $[0, T]$ with normalised pressure (a pressure that repeats with the flow). This holds under each of three readings of Tao's normalised pressure (repeating; repeating with mean zero; his equation (9) coefficient by coefficient) and whatever period $L > 0$ the solution has. In symbols, the strongest form proved:

$$\exists F \text{ smooth and 1-periodic: } \Big(\forall T > 0,\ \mathrm{PeriodicData}(0, F, T)\Big) \wedge \forall T \geq 1,\ \neg\, \exists u, p,$$

so that $\neg\, \mathrm{Conjecture18P}$, $\neg\, \mathrm{Conjecture18A}$ and $\neg\, \mathrm{Conjecture18B}$, and the same for a solution of any period.

Status: proved in Lean (Lean 4.34.1, only the standard axioms propext, Classical.choice and Quot.sound; nanoda accepted exports of 91,179 declarations for period 1 and 95,687 for any period). Two short written steps connect the Lean definitions to Tao's words (the next two entries). Not refereed. The unforced case, Conjecture 1.4 (Clay's (B) on the torus), stays open.

```
-- name: TaoConjecture18.conjecture18_counterexample
-- name: TaoConjecture18.not_conjecture18P
-- name: TaoConjecture18.not_conjecture18A
-- name: TaoConjecture18.not_conjecture18B
-- name: TaoConjecture18Period.not_conjecture18_any_period
-- file: lean/TaoConjecture18.lean
theorem conjecture18_counterexample :
    ∃ F : VelocityField,
      (∀ T : ℝ, 0 < T → PeriodicData (fun _ => 0) F T) ∧
      ∀ T : ℝ, 1 ≤ T → ¬ ∃ u p,
        SmoothPeriodicSolution (PeriodicPressure T) (fun _ => 0) F T u p
theorem not_conjecture18P : ¬ Conjecture18P
theorem not_conjecture18A : ¬ Conjecture18A
theorem not_conjecture18B : ¬ Conjecture18B
-- file: lean/TaoConjecture18Period.lean
theorem not_conjecture18_any_period : ¬ Conjecture18AnyPeriod
```

Proof: chapter 4 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-04-tao-2011); proofs/TAO-1-8-WRITTEN-STEPS.md; Lean theorems named in the code block above.

Lemma 17: smoothness on the closed slab
(Sonny, 2026; step 1 of the Conjecture 1.8 result)

Credit: This result (Sonny): proved (written).

Tao's smoothness of u and p on the closed slab $[0,T] \times R^3$ gives the smoothness relative to the slab that the Lean statement uses. The book calls this a standard fact (compare Seeley's extension theorem, 1964).

Status: written; read only inside this project. Not formalised in Lean.

Proof: proofs/TAO-1-8-WRITTEN-STEPS.md, "Step 1".

Lemma 18: Tao's pressure meets each reading
(Sonny, 2026; step 2 of the Conjecture 1.8 result)

Credit: This result (Sonny): proved (written).

Tao's normalised pressure is $p = -\Delta^{-1}\partial_i\partial_j(u_i u_j) + \Delta^{-1}\nabla \cdot f$. On the torus, Δ^{-1} returns a periodic function of mean zero. So a pressure given by Tao's formula

repeats with the period of the solution and has mean zero at each time; this covers the periodic and mean-zero readings, and it is all the result needs, because Lean proves that the coefficient reading implies the mean-zero reading, which implies the periodic reading. The coefficient reading is also met: with the convention $\hat{f}(k) = \int_{\text{cell}} e^{-2\pi i k \cdot x} f(x)\, dx$, for $k \neq 0$ the coefficient of Tao's pressure is $-\frac{k_i k_j}{|k|^2}(u_i u_j)\hat{}(k) + \frac{2\pi i k_i}{-4\pi^2 |k|^2}\hat{f}_i(k)$, and the coefficient at $k = 0$ is 0. The same holds at any period L.

Status: written; the coefficient calculation is a hand calculation, not formalised, and the result does not depend on it.

Proof: proofs/TAO-1-8-WRITTEN-STEPS.md, "Step 2" and "The same steps for a solution of any period".

Lemma 19: the shift lemma

Credit: Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (Lean).

Throughout this and the next entries: $\nu > 0$, f is smooth and 1-periodic in space, (u, p) is a smooth pair on $R^3 \times [0, \infty)$, and "from rest" means $u(\cdot, 0) = 0$. Class (iii-a), the erratum read literally with periodic pressure, is: (1), (2), (3), (11) and the weak identity in place of (10), plus periodic p. If (u, p) is in class (iii-a), so is $(u(\cdot + e_j), p)$, with the same p and f.

Status: proved in Lean and nanoda.

```
-- NSClose.shift_literal   (file lean/NSClose/Literal.lean)
```

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open), table of results under reading (iii-a); Lean theorem named in the code block above.

Lemma 20: the period-defect system

Credit: Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (Lean).

The period defect $\delta = u(\cdot + e_j) - u$ solves

$$\partial_t \delta + \Big(u(\cdot + e_j)\cdot\nabla\Big)\delta + (\delta\cdot\nabla)u - \nu\Delta\delta = 0, \qquad \operatorname{div}\delta = 0, \qquad \delta(\cdot,0) = 0.$$

There is no pressure term, because both pressures are the same function.

Status: proved in Lean and nanoda.

```
-- NSClose.defect_system   (file lean/NSClose/Literal.lean)
```

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open); Lean theorem named in the code block above.

Lemma 21: the nonzero-defect lemma

Credit: Tools: Lean 4; leanchecker; the nanoda kernel checker. This result (Sonny): proved (Lean).

If f has no competitor in class (ii) (the erratum as intended: (i) plus $p(x + e_j, t) = p(x, t)$; the forced breakdown with repeating pressure supplies such an f, with exactly this hypothesis), then every class-(iii-a) competitor for f has some j with $\delta \not\equiv 0$ solving the period-defect system.

Status: proved in Lean and nanoda. The bridge joining it directly to the forced breakdown with repeating pressure and to the comparator class of unrestricted-velocity competitors was checked by Lean and nanoda and rebuilt from source (standard axioms only; the replay tool leanchecker passed).

```
-- NSClose.competitor_carries_nonzero_defect   (file lean/NSClose/Literal.
-- NSCloseBridge.openai_force_literal_competitors   (file lean/NSClose/Bric
```

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open); Lean theorems named in the code block above.

Lemma 22: the subclass-exclusion lemma

Credit: Tools: Lean 4. This result (Sonny): proved (Lean).

If every literal-class competitor in a chosen subclass has zero period defects, and the force is excluded in the periodic class, that subclass has no competitor. The zero-defect hypothesis must be established separately.

Status: proved in Lean (standard axioms only, re-checked from copied source); the zero-defect hypothesis must be established separately.

```
-- NSClose.subclass_exclusion   (file lean/NSClose/Literal.lean)
```

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open); Lean theorem named in the code block above.

Theorem 23: at most linear growth forces periodicity, the linear-growth repetition theorem

Credit: Builds on: the classical weighted-energy uniqueness argument for parabolic problems in a growth class (Tychonoff). Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (Lean).

If (u,p) is in class (iii-a) and $|u(x,t)| \le M(1+|x|)$ on $R^3 \times [0,T]$, then $u(\cdot + e_j, t) = u(\cdot, t)$ on $[0,T]$ for each j.

Status: proved in Lean and nanoda (Lean 4.34.1, standard axioms). The written proof uses the weight $\exp(-\lambda(t)(1+|x|^2))$ with $\lambda(t) = e^{K(t-t_0)}$, $K = 24M + 4M^2/\nu + 32\nu$, cutoff $\chi(x/R)^2$, on steps of length $(\ln 2)/K$; the stretching term is integrated by parts using $\mathrm{div}\,\delta = 0$, so only the size of u enters, not its gradient. The Lean proof takes the same route with different constants: weight $\exp(-\lambda(t)(1+|x|^2))$ with $\lambda(t) = 1 + K(t-t_0)$, steps of length $1/K$, $K = 24M + 8M^2/\nu + 96\nu$; it uses only that the force repeats.

```
-- files lean/OWSuper.lean, OWLinear.lean and LinearGrowthCalc ... LinearGr
OriginalWording.linear_growth_repetition : OriginalWording.LinearGrowthRepe
```

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open); complete weighted estimate in proofs/CLOSURES.md, section 7; Lean theorem named in the code block above.

Theorem 24: for OpenAI's force, growth must be superlinear before time 1, the superlinear-growth theorem

Credit: Builds on: OpenAI's construction (2026) and its Corollary 10.6. Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (Lean).

Every class-(iii-a) competitor u for OpenAI's force satisfies

$$\sup_{R^3 \times [0,T]} \frac{|u(x,t)|}{1+|x|} = \infty \quad \text{for some } T < 1.$$

Proof in outline: otherwise the linear-growth repetition theorem makes u periodic on every $[0,T]$, $T<1$; at viscosity one, OpenAI's periodic uniqueness theorem on a closed slab gives $u = U$ on each closed slab before time 1; an application at another viscosity additionally needs the appropriate viscosity-scaling bridge; continuity of u on the compact set $[0,1]^3 \times [0,1]$ then contradicts $\limsup_{t \uparrow 1} \|U(t)\|_\infty = \infty$ (OpenAI, Corollary 10.6, p. 125).

Status: proved in Lean and nanoda. Corollary 10.6 enters as OpenAI's own Lean theorem, and the viscosity-scaling bridge is Lean.

```
OriginalWording.superlinear_growth
OriginalWording.superlinear_growth_of_linear_repetition   (the same, from t
```

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open); Lean theorems named in the code block above.

Proposition 25: the class (iii-a) has fast-growing members

Credit: Builds on: A. N. Tychonoff (1935), heat-equation non-uniqueness without growth bounds. This result (Sonny): proved (written).

At zero force, $u = (H(t, x_2), 0, 0)$, $p = 0$, with

$$H(t,y) = \sum_{n \geq 0} \frac{g^{(n)}(t)\, y^{2n}}{\nu^n (2n)!}, \qquad g(t) = \begin{cases} e^{-1/t^2} & t > 0, \\ 0 & t \leq 0, \end{cases}$$

is a smooth, nonzero, non-periodic class-(iii-a) solution from rest. Being non-periodic, it grows faster than linearly in x_2, by the linear-growth repetition theorem. The bound $\exp(c\, x_2^2/t)$ of the written argument is an upper bound, not a proved growth rate.

Status: written (complete argument: the heat-shear control in the literal-erratum proofs, section 5); not in Lean.

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open); proofs/LITERAL-ERRATUM.md, section 5.

Proposition 26: uniqueness already fails on a time slab for OpenAI's force

Credit: Builds on: OpenAI's construction (2026). Tools: Lean 4. This result (Sonny): proved (written).

OpenAI's force and solution vanish on $[0, t_0]$, $t_0 = 1 - \ell^{-2} \in (0, 1)$, where $\ell > 1$ is a parameter of their construction. On that slab the solution of the fast-growing members result is a competitor to $U = 0$.

Status: written (the positive initial interval on which the force is zero is Lean; the fast-growing members result is written).

```
-- initial zero-force interval: lean/InitialGap.lean
--   InitialGap.periodic_zero_mean_force_with_gap
```

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open); proofs/LITERAL-ERRATUM.md.

Proposition 27: the additive shear, a counterexample to the strong form

Credit: This result (Sonny): proved (written).

Chapter 6 asks about competitors of the form $u = U + H(t, \langle k, x \rangle)d$ with $\langle k, d \rangle = 0$, where U is a periodic solution. The strong form of its statement is: $U(t, x) = 0$ wherever some period difference of $H(t, \cdot)$ is nonzero at $\langle k, x \rangle$. For a general periodic U the strong form is false: with force e_3, the periodic solution $U = te_3$ (pressure 0) and the heat shear H of the fast-growing members result, $u = te_3 + H(t, x_2)e_1$ solves the equations from rest with pressure 0, while $U \neq 0$ at every point with $t > 0$ and $H(t, \cdot)$ has nonzero period differences at arbitrarily small times.

Status: proved in writing (a second reading within this project); checked numerically; not in Lean.

Proof: proofs/ADDITIVE-SHEAR.md; chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open).

Proposition 28: the additive shear, the weaker statement that holds for every periodic U

Credit: This result (Sonny): proved (written).

Let $\nu > 0$, and let (U, P) and (u, p) solve the same equations with the same force, with U and P periodic and $p(t, x + e_j) = p(t, x)$, and $u = U + H(t, \langle k, x \rangle)d$ with $\langle k, d \rangle = 0$, $d \neq 0$. Then wherever a period difference of $H(t, \cdot)$ is nonzero at $\langle k, x \rangle$, the part of $(d \cdot \nabla)U(t, x)$ orthogonal to d is zero. This needs only that both solutions have the same force and repeating pressures; it uses neither the heat equation for H nor the start from rest.

Status: proved in writing (a second reading within this project); not in Lean.

Proof: proofs/ADDITIVE-SHEAR.md; chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open).

Computation 29: the erratum's sign pattern and the defect identities, computed

Credit: This result (Sonny): computed.

The erratum's sign pattern was checked numerically against an exact smooth solution: with the erratum's signs the pointwise defect is at round-off level, and with the printed signs of (12) it is of order one. The defect identity for additive shear competitors $u = U + H(t, \langle k, x \rangle)d$, and the linear-plus-periodic identity $(A' + A^2)e_j + (Ae_j \cdot \nabla)U = 0$, were checked to round-off (relative error of order 10^{-14}) with analytic derivatives.

Status: computed.

Proof: chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open), section "Computed".

Lemma 30: the smooth-solutions identity

Credit: The chapter's credit section assigns the lemmas of section 5 other than the curl of OpenAI's force to Sonny. This result: Sonny.

If (u, p) is smooth for $t > 0$ and satisfies (1), (2) there, the erratum's identity holds for every test of the text:

$$-\iint u \cdot \partial_t \theta - \sum_{ij} \iint u_i u_j \, \partial_{x_j} \theta_i = \nu \iint u \cdot \Delta\theta + \iint f \cdot \theta + \iint p \, \mathrm{div}\theta,$$

for every smooth vector field θ compactly supported in $R^3 \times (0, \infty)$. Requiring the identity in place of (10), or beside it, therefore adds nothing for smooth solutions.

Status: written; Lean, nanoda, for solutions smooth up to $t = 0$ with any smooth force, with the integrability conditions of the weak class; such solutions, and those smooth only for $t > 0$, lie in the trace weak class.

```
-- smooth-solutions identity (solutions smooth up to t = 0; file lean/WRBr
--   WrittenRows.smooth_solutions_identity, traceWeak_of_smooth
--   WrittenRows.traceWeak_of_positive   (solutions smooth only for t > 0;
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The smooth-solutions identity".

Lemma 31: the extra-identity reduction

Credit: Tools: Lean 4. This result (Sonny): divergence-free-test, zero-extension, initial-trace and one-cell forms proved (written).

Write $A = \iint u \cdot \partial_t \theta$, $B = \sum \iint u_i u_j \, \partial_j \theta_i$, $L = \nu \iint u \cdot \Delta\theta$, $F = \iint f \cdot \theta$ and $P = \iint p \mathrm{div} \theta$. By the smooth-solutions identity, $-A - B = L + F + P$. So, with the text's tests: (a) printed (12), $A - B = L + F - P$, holds for all θ iff $A + P = -\iint (\partial_t u + \nabla p) \cdot \theta = 0$ for all θ, that is, iff $\partial_t u + \nabla p = 0$; (b) with only the time sign corrected, $-A - B = L + F - P$ iff $P = 0$, that is, iff $\nabla p = 0$; (c) with only the pressure sign corrected, $A - B = L + F + P$ iff $A = 0$, that is, iff $\partial_t u = 0$. Other test conventions: with divergence-free tests only, P vanishes, so (a) and (c) become $\mathrm{curl} \partial_t u = 0$ and (b) holds for every solution. With tests reaching $t = 0$ and zero extension, each identity gains the term $\int u(x,0) \cdot \theta(x,0) \, dx$, and the erratum's identity then also forces $u(\cdot, 0) = 0$; tests supported in $t > 0$ still give (a) to (c). With periodic tests on one cell, see the cell-test detection lemma; for non-periodic u, tests supported inside the open cell give (a) to (c) on that cell only.

Status: written; the quantified test identities (a) to (c) are Lean; for smooth solutions and the text's test fields the reduction itself (the four integrations by parts, the fundamental lemma and the step to $t = 0$), for any smooth force, is Lean. Divergence-free-test, zero-extension, initial-trace and one-cell forms: written.

```
-- extra-identity reduction
--   NSContinuation.Wprime_printed_test_iff, Wprime_time_corrected_test_if
--   module NSLemmas.IdentityAlgebra
--   reduction_printed_force, reduction_time_corrected_force,
--     reduction_pressure_corrected_force, fourIntegrals, eq_zero_of_integ
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The extra-identity reduction".

Lemma 32: the curl of OpenAI's force

Credit: Builds on: OpenAI's construction (2026). Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (written).

OpenAI's force f satisfies $\operatorname{curl} f \not\equiv 0$. Proof, as written: suppose $\operatorname{curl} f \equiv 0$; let $m(t)$ be the cell mean of f and put $g = f - m$; define $\varphi(x,t)$ as the line integral of g from 0 to x; then φ is well defined, smooth up to $t = 0$ and periodic, so $u(x,t) = \int_0^t m$, $p = \varphi$ is smooth, periodic in velocity and pressure, starts from rest, and solves (1), (2), which contradicts the forced breakdown with repeating pressure.

Status: written; Lean, nanoda, for every viscosity (the periodic-potential premise of the conditional Lean form is proved for forces smooth on all of space-time, which OpenAI's force is).

```
-- curl of OpenAI's force (no premise left; files lean/WRCurl.lean, WRUnifo
--   WrittenRows.openai_force_curl_nonzero, potential_of_curl_free, openai_
--   NSContinuation.uniformPrimitive_solution   (the uniform-mean solution)
--   NSContinuation.H_conditional_on_periodic_potential   (the conditional
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The curl of OpenAI's force".

Lemma 33: the frozen-vorticity obstruction

Credit: Tools: Lean 4. This result (Sonny): proved (written).

Let f be smooth with $\operatorname{curl} f \not\equiv 0$, and let (u,p) start from rest. Any one of the following conditions rules out a solution, with any velocity or pressure condition: (i) $\partial_t u = 0$ (then $u \equiv 0$, so $\nabla p = f$, which is curl-free); (ii) $\partial_t u + \nabla p = 0$ or $\operatorname{curl} \partial_t u = 0$ (then $\partial_t \omega = 0$ for $\omega = \operatorname{curl} u$, so $\omega \equiv \omega(\cdot, 0) = 0$;

a curl-free, divergence-free u has $\Delta u = 0$ and $(u \cdot \nabla)u = \frac{1}{2}\nabla|u|^2$; so (1) gives $\mathrm{curl} f = \mathrm{curl}\partial_t u = 0$). On one cell, the same holds on that cell, and periodicity of f carries $\mathrm{curl} f = 0$ everywhere. Weak form: $\partial_t \omega = 0$ in distributions, and weak continuity at $t = 0$ gives $\omega \equiv 0$; then u is harmonic in x (Weyl's lemma) and the same identities follow. With the curl of OpenAI's force, this decides every degenerate reading for OpenAI's force, except $\nabla p = 0$ with velocity periodicity dropped (open; see the reduced table).

Status: written; the smooth case is Lean (all three degeneracies, from rest, with no periodicity assumption). Weak form in the trace weak class: Lean; weaker traces: written.

```
-- frozen-vorticity obstruction
--   NSContinuation.D1_smooth
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The frozen-vorticity obstruction".

Lemma 34: the shear-start obstruction

Credit: Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (written).

Explicit starts with no solution. (i) $\partial_t u = 0$: the shear $u^\circ = (\sin 2\pi x_2, 0, 0)$ gives $u \equiv u^\circ$; then $(u^\circ \cdot \nabla)u^\circ - \nu\Delta u^\circ = 4\pi^2 \nu u^\circ$ would have to be a gradient, but its curl is nonzero. (ii) $\partial_t u + \nabla p = 0$: for the same shear, subtracting this condition from (1) leaves $(u \cdot \nabla)u - \nu\Delta u = 0$; at $t = 0$ this is $4\pi^2 \nu u^\circ \neq 0$. (iii) $\mathrm{curl}\partial_t u = 0$: the curl of (1) at $t = 0$ gives $\nu\Delta\omega^\circ = \mathrm{curl}((u^\circ \cdot \nabla)u^\circ) = 0$, but $\Delta\omega^\circ = -4\pi^2\omega^\circ \neq 0$. (iv) $\nabla p = 0$: take the divergence of (1) at $t = 0$; for $u^\circ = (\sin 2\pi x_2, \sin 2\pi x_1, 0)$ the term $\partial_i\partial_j(u_i^\circ u_j^\circ) = 8\pi^2 \cos 2\pi x_1 \cos 2\pi x_2$ would have to vanish. On R^3, the swirl $u^\circ = (-x_2, x_1, 0)e^{-|x|^2}$ does the same in each case: each quantity is nonzero at sample points. Smooth for $t > 0$ only: the same conclusions hold, because u is continuous at $t = 0$ and the quantities above converge in distributions. Weak forms: for $\partial_t u = 0$, $u \equiv u^\circ$ is smooth; for $\partial_t u + \nabla p = 0$ or $\mathrm{curl}\partial_t u = 0$ with periodic u, the difference $u - u^\circ$ is a

periodic harmonic gradient, hence a constant $c(t)$, and $((u^\circ + c)\cdot\nabla)u^\circ = \nu\Delta u^\circ$ fails for the shear.

Status: written; the four periodic obstructions are Lean (stationary shear; time-plus-pressure shear; curl-time shear; crossed shear with zero pressure gradient, every viscosity). Whole-space form: Lean, with a compactly supported start equal to the shear on the unit ball in place of the swirl (the claim is the same: some start has no solution). Positive-time-only form and divergence-free-test form: Lean, nanoda. Weak forms: Lean in the trace weak class; weaker traces: written. Values checked numerically.

```
-- shear-start obstruction
--   D2_shear_stationary, D2_shear_cancel, D2_shear_curl_time, D2_crossed_z
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The shear-start obstruction".

Lemma 35: the witness forces for (D)

Credit: Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (written).

Witnesses for the statement (D). (i) $\partial_t u = 0$ or $\nabla p = 0$: with $\partial_t u = 0$, take $u^\circ = 0$ and $f = (e^{-t}\sin 2\pi x_2, 0, 0)$, whose curl is nonzero; with $\nabla p = 0$, take $u^\circ = 0$ and $f = e^{-t}\nabla\cos 2\pi x_1$, then $\operatorname{div} f(\cdot, 0) \neq 0$ but $\partial_i\partial_j(u_i u_j) \to 0$ as $t \to 0$. Under the divergence-free force convention, or with the force compactly supported in time, take instead the primitive start $u^\circ = (\sin 2\pi x_2 + \sin 2\pi x_3, \sin 2\pi x_1, 0)$ with $f = 0$, which is divergence-free and compactly supported in time; it rules out both $\partial_t u = 0$ and $\nabla p = 0$ at a point inside the cell, for every viscosity. (ii) Smooth zero extension: all derivatives vanish at $t = 0$, so (1) forces $f(\cdot, 0) = 0$; any force with $f(\cdot, 0) \neq 0$ has no solution. (iii) Zero mean velocity for all t: the start $u^\circ = e_1$ is smooth, divergence-free and periodic, and violates the condition at $t = 0$; under "minimal period 1", use $(1 + \sin 2\pi x_2 + \sin 2\pi x_3, \sin 2\pi x_1, 0)$. These forces satisfy (8), (9), because every derivative decays like e^{-t}.

Status: written; the smooth-zero-extension case, the ordinary case of the zero-mean case and the force admissibility of the stationary case are Lean; the zero-mean case with starts of least period one, and with a zero-mean start, is Lean, nanoda. The exponential shear force is not compactly supported in time (Lean), so the compact-time readings use the primitive start with zero force.

```
-- witness forces for (D)
--   forceProfile_admissible, D3a_shear_stationary_exclusion, D3a_gradient_
--   exponential_witness_not_compact_time
--   NSLemmas.D3b_zero_extension_force_zero
--   NSContinuation.D3b_admissible_witness_exclusion
--   NSLemmas.D3c_nonzero_mean_start, D3c_e1_start
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The witness forces for (D)"; proofs/CLOSURES.md, section 3.

Lemma 36: the whole-space energy exclusion

Credit: Tools: Lean 4. This result (Sonny): proved (written).

Let f be smooth and periodic with $\mathrm{curl} f \not\equiv 0$. Then no (u,p), smooth or weak, has $\sup_t \int_{R^3} |u|^2 < \infty$ and satisfies the erratum's identity. With the curl of OpenAI's force this applies to OpenAI's force. In the periodic class, finite energy on R^3 forces $u \equiv 0$, so a nonzero (B) start is excluded. (The written argument takes the test $\theta = \psi(t)\mathrm{curl}(\varphi(x/R)\, g(x))$ with $g = \mathrm{curl} f(\cdot, t_1)$; the velocity terms are $O(R^{3/2})$, the force term is $cR^3 + O(R^2)$ with $c \neq 0$.)

Status: written; the growth contradiction is Lean, and the whole is Lean conditional on its cutoff estimates.

```
-- whole-space energy exclusion
--   NSContinuation.E_growth_contradiction, E_conditional_on_cutoff_estimat
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The whole-space energy exclusion".

Lemma 37: tilt removal while the force is zero

Credit: Builds on: OpenAI's construction (2026). Tools: Lean 4. This result (Sonny): proved (written).

OpenAI's force vanishes for $0 \le t \le t_0$ for some $t_0 > 0$ (Lean: the construction keeps a positive initial interval on which its force is zero). Then OpenAI's force has a printed-class competitor (u, p) from rest with $u \equiv 0$ and $p \equiv 0$ for $t \in [0, \varepsilon]$. In particular its pressure is periodic at $t = 0$, and its zero extension is C^∞. (Steps: remove the tilt near $t = 0$ with $u_Z(x,t) = u(x - Z, t) + Z'(t)$ and $p_Z = p(x - Z, t) - Z''(t) \cdot x$ for a smooth Z with $Z = \chi(t) \int_0^t \int_0^s a$, $\chi \equiv 1$ on $[0, \varepsilon]$ and $\chi \equiv 0$ for $t \ge t_0/2$; check it is still a competitor; on $[0, \varepsilon]$ the pair is periodic in velocity and pressure with zero force, so $\frac{d}{dt}\|u_Z\|_{\text{cell}}^2 = -2\nu\|\nabla u_Z\|^2 \le 0$ and $u_Z \equiv 0$.)

Status: written; OpenAI's zero window, gauge removal and zero extension are Lean; the whole is Lean conditional on the translated-zero-window premise (steps 1 to 3 of the written proof).

```
-- tilt removal
--   gauge_preserves_printed, zeroPast_smooth_of_zero_window, G_conditional
--   InitialGap.periodic_zero_mean_force_with_gap   (the construction's zer
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "Tilt removal while the force is zero".

Lemma 38: the positive-time exclusion

Credit: Builds on: OpenAI's construction (2026) and its Corollary 10.6. Tools: Lean 4. This result (Sonny): proved (written).

The exclusion of the forced breakdown with repeating pressure also holds when the competitor is C^∞ only for $t > 0$ and continuous on $R^3 \times [0, \infty)$. Proof as written: let U be OpenAI's solution, smooth on $[0, 1)$ and periodic with $\|U(t)\|_\infty$ unbounded as $t \uparrow 1$; put $w = u - U$; periodic integration removes the transport and pressure terms, so $\frac{d}{dt}\|w\|^2 \le 2\sup_{[0,T]}\|\nabla U\|_\infty \|w\|^2$ on $(0, T]$ for every $T < 1$; uniform continuity on the compact cell gives $\|w(s)\| \to 0$ as $s \to 0$, so $w \equiv 0$ on $[0, 1)$; but u is continuous on $[0, 1]^3 \times [0, 1]$, so it is bounded there, which contradicts the blowup of U.

Status: written; the general positive-time exclusion is Lean; the binding to OpenAI's exact force is written. Inputs: OpenAI's Lean candidate properties and Corollary 10.6.

```
-- positive-time exclusion
--   NSLemmas.PositiveTime.S_exclusion
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The positive-time exclusion".

Lemma 39: the coarser-period reduction

Credit: Tools: Lean 4. This result (Sonny): shift uniqueness on the N torus proved (written).

Let N be a positive integer, and keep the same 1-periodic force and the same periodic start. If (u, p) is a competitor whose velocity and pressure have period N (lattice NZ^3), then it is 1-periodic. Proof as written: $(u(\cdot + e_j), p(\cdot + e_j))$ is also a competitor, since f is 1-periodic; the energy argument on the NZ^3 torus makes the two equal; so u is 1-periodic, and $p(x + e_j) - p(x) = c_j(t)$; N-periodicity of p gives $Nc_j = 0$. The forced breakdown with repeating pressure then applies. With periodic velocity, the same shift argument shows that printed (B) with period-N solutions is the same as with period 1. ("Period N with N unspecified" is not enough.)

Status: written, partly Lean: finite period kills the jump and pressure has unit period: Lean. Shift uniqueness on the N torus: written.

```
-- coarser-period reduction
--   NSLemmas.N_finite_period_kills_jump, N_pressure_unit_period
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The coarser-period reduction".

Lemma 40: the pressure-periodicity lemma

Credit: Tools: Lean 4. This result (Sonny): proved (Lean).

(a) If u is periodic, then $\nabla p = f - \partial_t u - (u \cdot \nabla)u + \nu \Delta u$ is periodic. So "pressure modulo affine functions" adds nothing to the printed class. (b) In that case $p(x + e_j, t) - p(x, t) = c_j(t)$. If p is sublinear in x at each t, then $c_j = 0$, so p is periodic.

Status: Lean.

```
-- pressure-periodicity lemma
--   NSLemmas.Q_gradient_periodic, Q_pressure_periodic, Q_printed_sublinear
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The pressure-periodicity lemma".

Lemma 41: the cell-test detection lemma

Credit: Tools: Lean 4. This result (Sonny): proved (written).

For periodic u and spatially periodic tests on the unit cell, the boundary terms of the pressure integral leave the tilt term $\tau(\theta) = \sum_j a_j \int_{\text{face}_j} \theta_j$. Each identity then holds iff its condition from the extra-identity reduction holds and $a \equiv 0$. Tests supported inside the cell give that condition, and then $\tau = 0$ for all θ gives $a = 0$. So the pressure is periodic.

Status: written; the finite linear detection step is Lean; face integrals and test duality: written.

```
-- cell-test detection lemma
--   NSLemmas.C4_detect_all_tilts
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The cell-test detection lemma".

Lemma 42: the divergence-free force reduction

Credit: Tools: Lean 4. This result (Sonny): proved (written).

On the torus only. Let f be smooth up to $t = 0$ and satisfy (8), (9). Let $\varphi(\cdot, t)$ be the mean-zero periodic solution of $\Delta\varphi = \mathrm{div} f(\cdot, t)$, and put $f' = f - \nabla\varphi$. Then f' is smooth, periodic, divergence-free and satisfies (8), (9) (Fourier bounds); it has the same curl and the same cell mean as f; it is zero at every time at which f is identically zero. Further, (u, p) solves (1), (2) with f iff $(u, p - \varphi)$ solves them with f'. This map keeps every velocity condition and every smoothness reading; it keeps every pressure reading except $\nabla p = 0$ (repeating at all times, up to a function of t, modulo affine functions, at $t = 0$ only, sublinear, at most linear, bounded); $\nabla p = 0$ carries over one way only: if $\nabla(p - \varphi) = 0$, then p repeats at all times. For $\mathrm{div} u = 0$ the gradient work on a cell is $\int_Q \nabla\varphi \cdot u = \oint_{\partial Q} \varphi\, u \cdot n$; it vanishes when the velocity is periodic, so the cell energy equality and inequality carry over for periodic velocity (nothing is claimed for free velocity). Two identities add the pressure term (the erratum's, and the one with only the pressure sign corrected) and carry over; two subtract it (printed (12), and the one with only the time sign corrected), change by $-2 \iint \nabla\varphi \cdot \theta$, and do not carry over except on divergence-free tests. On all of space the projection breaks (5); the lemma applies on the torus.

Status: proved in writing (a second reading within this project); the residual, printed-solution and pressure-periodicity equivalences are Lean; torus Poisson existence and bounds, and the pressure, energy and identity transfers: written.

```
-- divergence-free force reduction
--   NSLemmas.Helm_residual_invariant, Helm_printed_solution_iff, Helm_pres
```

Proof: proofs/DIVERGENCE-FREE-FORCE.md; chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5.

Lemma 43: the equivalence lemma for (B)

Credit: Builds on: Terence Tao (2011, p. 6 and Lemma 4.1). Tools: Lean 4; the nanoda kernel checker; lean4lean. This result (Sonny): proved (written).

For (B) with periodic flow: (i) printed existence for u° is equivalent to existence with periodic pressure (the Lean equivalence, built in the book's default target; checked by nanoda and lean4lean), so the pressure-at-$t = 0$-only, modulo-affine, sublinear and linear pressure readings give the same question; (ii) a start with mean c reduces to mean 0 by the Galilean change $v(x,t) = u(x + ct, t) - c$, which keeps both periods; (iii) the periodic-pressure solution satisfies the energy equality, so the cell-energy and energy-inequality readings give the same question.

Status: written; partly Lean, using the equivalence of printed and periodic-pressure existence.

```
-- equivalence lemma for (B)
--   NSLemmas.M_printed_periodic_pressure, M_intermediate_existence
--   JustExploringInherited.inherited_existence_iff  (file InheritedBridge
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "The equivalence lemma for (B)".

Lemma 44: the pressure-on-all-of-space lemma

Credit: Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (Lean).

Apply the pressure sentence to all of space. If $p(\cdot,0)$ is periodic (or $\nabla p(\cdot,0)$ is) and u° satisfies (4), then $-\Delta p(\cdot,0) = \partial_i\partial_j(u_i^\circ u_j^\circ) - \operatorname{div} f(\cdot,0)$; the left side is periodic, even if only $\nabla p(\cdot,0)$ is; the right side tends to zero at infinity, by (4) and (5); a periodic function or distribution that tends to zero at infinity in this sense is zero; so $\partial_i\partial_j(u_i^\circ u_j^\circ) = \operatorname{div} f(\cdot,0)$. The swirl start violates this with $f = 0$ (its left side is -2 at the origin), and any force with $\operatorname{div} f(\cdot,0) \neq 0$ violates it from rest.

Status: Lean, nanoda for solutions smooth up to $t = 0$ (every pressure reading) and for solutions smooth for $t > 0$ with the pressure repeating at every positive time; written for smoothness for $t > 0$ only with the pressure repeating up to a constant or modulo affine functions. The Poisson identity and the decay are hypotheses, not derived from the data assumptions.

```
-- pressure-on-all-of-space lemma
--   NSLemmas.AP_periodic_poisson_source_zero, periodic_decays_zero
--   second-round route (file lean/WRPressure.lean; positive-time share in
--   WrittenRows.no_solution_cross_lap_periodic, A_pressure_all_times_not_
--     A_pressure_gradient_periodic_not_met, A_pressure_at_zero_not_met, A_
```

Proof: proofs/CLOSURES.md, section 4; chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5.

Theorem 45: Seeley's extension theorem (Seeley, 1964, cited)

Credit: R. T. Seeley, *Extension of C^infinity functions defined in a half space*, Proc. AMS 15 (1964). Cited.

"C^∞ up to $t = 0$" equals "restriction of a C^∞ function". Both readings of smoothness are therefore one class, Lean's notion of smooth on the closed half-space.

Status: cited (not rechecked here).

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-

every-reading), section 5, "Cited".

Theorem 46: global weak solutions (Leray, 1934; Hopf, 1951, cited)

Credit: J. Leray (1934) and E. Hopf (1951). Cited.

Global weak solutions exist. Fefferman's own page 4 states the whole-space case. Leray's global whole-space weak existence theorem applies to divergence-free L^2 initial data and integrable-in-time L^2 forces.

Status: cited (not rechecked here).

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "Cited"; proofs/CLOSURES.md, section 5.

Proposition 47: the identity algebra of the trace weak class

Credit: Tools: Lean 4. This result (Sonny): proved (written).

Call the trace weak class the class of distributional solutions of $\partial_t u + \operatorname{div}(u \otimes u) - \nu\Delta u + \nabla p = f$, $\operatorname{div} u = 0$ on positive times, with u locally square integrable, the products locally integrable, and $u(t) \to u^\circ$ strongly in $L^2(K)$, for every compact spatial K, as $t \downarrow 0$ along the full-measure time representative. Every competitor in this class satisfies the corrected identity $-A - B = L + F + P$ (letters as in the extra-identity reduction). Combining it with the printed or half-corrected identity gives, respectively, $A + P = 0$, $P = 0$ and $A = 0$. For arbitrary compact vector tests these are exactly $\partial_t u + \nabla p = 0$, $\nabla p = 0$ and $\partial_t u = 0$ in distributions; with divergence-free tests the two time conditions give $\operatorname{curl}\partial_t u = 0$ and the pressure condition adds nothing. For a binding identity tested through zero after zero extension, the time term has coefficient $+1$ or -1 in every sign choice, and a direct trace argument gives $\int u^\circ \cdot \varphi = 0$ for every φ, so $u^\circ = 0$.

Status: written; the scalar sign identities, uniqueness of the pairing limit, and the limiting compatibility implication are Lean in the Lean close-readings module; the distributional differentiation, integral estimates and PDE-to-pairing bridges are written.

Proof: proofs/CLOSURES.md, section 2; formal scope in proofs/CLOSURES-FORMAL-SCOPE.md.

Theorem 48: one primitive periodic start settles the weak degenerate-identity groups of (D) and (B)

Credit: Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (written).

Put $k = 2\pi$ and $v(x,y,z) = (\sin(ky) + \sin(kz),\ \sin(kx),\ 0)$, $f = 0$. This start is smooth, divergence free, mean zero on a cell and unit periodic, with full period lattice Z^3. Direct differentiation gives
$\mathrm{divdiv}(v \otimes v) = 2k^2\cos(kx)\cos(ky)$,
$\omega = \mathrm{curl}v = (0,\ k\cos(kz),\ k\cos(kx) - k\cos(ky))$ and $\Delta\omega = -k^2\omega$; the second coordinate of the left side of the frozen-curl compatibility equation is $\nu k^3\cos(kz)$, positive at $(1/8, 1/8, 1/8)$ for every $\nu > 0$, and the left side of the pressure-free compatibility equation is $k^2 > 0$. Every remaining degenerate identity in the weak degenerate-identity groups of (D) and (B) forces one of these two equations, so this single start has no competitor. Consequently the weak degenerate-identity group of (D) is answered: yes and that of (B) is answered: no, in the trace weak class, under both printed and corrected identity combinations that actually reach these groups. The zero force works for every ν.

Status: written; checked in Lean for the trace weak class (the degenerate identities on the text's test fields, the four weak degenerate identities stated directly, the identity tested through time zero). Every axiom list is only propext, Classical.choice and Quot.sound, and nanoda accepted an export of 50,100 declarations containing the 82 final theorems. The conclusion uses the initial trace stated in section 1 of the proof file.

```
-- weak solutions (files lean/WCCore.lean ... WCFree.lean; build target Rea
--   the trace weak class and Clay's (A) to (D) with weak solutions: module
--   statement-level results: modules WCMain, WCPressure, WCFree
--   WeakClass.B_weak_degenerate_not_met_min, D_weak_degenerate_met_min, B_
--     D_weak_zero_extended_met, A_weak_degenerate_not_met, C_weak_degenera
--     A_weak_zero_extended_not_met, C_weak_zero_extended_met,
--     A_weak_periodic_pressure_not_met, C_weak_periodic_pressure_met
```

Proof: proofs/CLOSURES.md, section 3; chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), "Written results behind section 4".

Theorem 49: a decaying start and the periodic-pressure obstruction

Credit: Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (written).

Let $a(x,y,z) = e^{-x^2 - y^2 - z^2}$, $s = (-ya,\ xa,\ 0)$ and $f = 0$. Then s satisfies (4) (all derivatives), f satisfies every version of (5), and $\mathrm{divdiv}(s \otimes s) = -2$, $\mathrm{curldiv}(s \otimes s) = 0$ and $\Delta \mathrm{curl} s = (0, 0, -20)$ at the origin, so the pressure-free and frozen-curl compatibility equations fail for every $\nu > 0$. A second obstruction needs no degenerate identity: if $p(0)$ or $\nabla p(0)$ has the prescribed spatial periods and $p(t) \to p(0)$ in spatial distributions, then $-\Delta p(0) = \mathrm{divdiv}(s \otimes s)$, whose left side is periodic and whose right side is a Schwartz function with value -2 at the origin; a periodic function tending to zero at infinity is identically zero, a contradiction. Consequences in the trace weak class: the degenerate and binding zero-extension subclasses of the (A) group (weak solutions, erratum applied to all of space) are answered: no and those of the corresponding (C) group are answered: yes; the all-positive-time periodic-pressure and periodic-gradient subclasses have the same answers; if a genuine periodic pressure trace at zero is required, the corresponding $t = 0$-only subclasses have those answers as well. A single unforced Schwartz start witnesses all these exclusions for (C); no property of OpenAI's force is needed.

Status: written; Lean in the trace weak class for a cut-off copy of the primitive start (not the swirl) and for a pressure that repeats at every

positive time with the energy bound uniform in time (nanoda: 50,100 declarations).

Proof: proofs/CLOSURES.md, section 4.

Theorem 50: the pressure at time zero only, with smoothness only afterwards, for (A)

Credit: This result (Sonny): proved (written).

Take the erratum applied to all alternatives, the pressure sentence "at $t = 0$ only", and smoothness "C^∞ for $t > 0$, continuous at $t = 0$". For (A), with $f = 0$ and u° any divergence-free datum satisfying (4), choose the Schwartz swirl s of the previous entry. A competitor would give $-\Delta p(0) = \mathrm{divdiv}(s \otimes s)$ in spatial distributions, with $p(0)$ unit periodic; its Laplacian is periodic, the right side is Schwartz, so periodicity forces it to vanish, but its value at the origin is -2. Contradiction. This group is answered: no for all 1,769,472 readings assigned to it. The argument needs neither a pointwise PDE at zero nor global derivative bounds nor the bound in (7), and it survives any additional identity or energy condition in those combinations.

Status: written.

Proof: proofs/CLOSURES.md, section 8.

Proposition 51: weak pressure at zero and one-cell identities

Credit: Builds on: Leray's global whole-space weak existence theorem (1934), cited. This result (Sonny): proved (written).

If weak p is specified only almost everywhere on $t > 0$, replace its representative at $t = 0$ by zero: every positive-time distribution and every text test integral is unchanged, so, with no other binding extra condition, existence with the pressure periodic "at $t = 0$ only" is equivalent to ordinary weak existence. Accordingly the pure, untraced $t = 0$-pressure subclasses give (A) answered: yes and (C) answered: no under those specified weak energy readings; if continuity of p at zero is required instead, the previous entries

give (A) answered: no and (C) answered: yes. A value and a trace are different mathematical conditions. For the one-cell identity: the extra corrected identity for every spatially periodic cell test is precisely the momentum equation for the periodic extensions of the restrictions to the open unit cube; it forces the velocity traces on opposite faces to agree. For Leray-Hopf competitors the one-cell corrected-identity subclasses give (A) answered: no and (C) answered: yes; for the broader trace weak class this refinement is not asserted.

Status: written; cited refinement (Leray-Hopf class, weak-strong uniqueness).

Proof: proofs/CLOSURES.md, section 5.

Proposition 52: OpenAI's construction with weak solutions and pressure freedom

Credit: Builds on: OpenAI's construction (2026). This result (Sonny): proved (written).

For $\nabla p = 0$ on all of space, the weak momentum equation is exactly the pressure-free Burgers system $\partial_t u + \operatorname{div}(u \otimes u) - \nu \Delta u = f$, $\operatorname{div} u = 0$, $u(0) = 0$; conversely any solution of it yields a pressure-free competitor with $p = 0$. In the nondegenerate branch with periodic velocity, strong pressure and zero mean velocity, mean balance gives $\frac{d}{dt} \operatorname{mean} u = \operatorname{mean} f$; so the fixed force meets (D) in this weak zero-mean branch iff $\operatorname{mean} f$ is not identically zero (the direction asserting weak existence is cited Leray-Hopf existence). For a periodic Leray-Hopf competitor with $\nabla p = 0$ everywhere, weak-strong comparison identifies it with the smooth periodic solution U of the unconstrained viscous Burgers equation, which is global; $(U, 0)$ is then a global smooth periodic Navier-Stokes competitor, contradicting the forced breakdown with repeating pressure: the corresponding subclass is answered: yes for Leray-Hopf competitors, for each force supplied by the forced breakdown with repeating pressure at its chosen viscosity (and for a coarser solution period N likewise). The group as a whole is open.

Status: written, with cited Leray-Hopf existence. This does not prove that every distributional competitor in the trace weak class satisfies the relative-

energy inequality; it does not apply when velocity periodicity is dropped or when ∇p vanishes only in a cell.

Proof: proofs/CLOSURES.md, section 6.

Proposition 53: the minimal-period (D) group, primitive starts

Credit: This result (Sonny): proved (written).

For (D) with a start of minimal period one: the degenerate branch ($\partial_t u + \nabla p = 0$ and $\operatorname{curl}\partial_t u = 0$) is answered: yes by the primitive start v with $f = 0$ (every ν). For the periodic-pressure branch, for one viscosity or data chosen for each viscosity, choose the primitive mean-zero start v, a preparatory force $F_{\text{pre}} = \partial_t V + \operatorname{div}(V \otimes V) - \nu\Delta V + \nabla P$ with $V(t) = \beta(t)v$ on $0 \le t \le 2$ and $-\Delta P = \operatorname{divdiv}(V \otimes V)$, joined to a delayed copy of the periodic-corollary force; any periodic-velocity, periodic-pressure competitor equals (V, P) on the preparatory interval, is at rest at time 2, and would contradict the forced breakdown with repeating pressure for the later force. These readings are answered: yes. Force conventions (divergence-free, zero mean, compact support in time) are preserved. The preparatory force depends on ν, so the proof gives the viscosity readings "data chosen for each ν" and "one ν" and no single force for all ν.

Status: written. (D) with a start of minimal period one is answered: yes for 13,148,160 of its 21,934,080 readings. The exact residual reductions for a periodic velocity and free pressure and for unrestricted velocity are reductions, not decisions; they stay open.

Proof: proofs/CLOSURES.md, section 7.

Theorem 54: the construction's force has exactly zero mean at every time

Credit: Tools: Lean 4. This result (Sonny): proved (written).

For every positive viscosity ν, the construction's force has exactly zero spatial mean at every time: the theorem supplies a construction witness (U, P, f, K), retains OpenAI's candidate properties and their exclusion of a global

periodic-velocity, periodic-pressure competitor at that same ν, and proves $\int_{\text{cell}} f(x,t)\,dx = 0$ for all t. A Lean result also proves a strictly positive initial interval on which the force vanishes identically. The reading "zero mean force" is answered: yes. The zero-mean-velocity conclusion for periodic velocity: with zero force mean, mean balance is $m' = -a$; if $m = 0$ for all time, $a = 0$ and the pressure is periodic, so the forced breakdown with repeating pressure excludes the competitor at the viscosity for which the force was constructed. It does not assert the same exclusion for a velocity unrestricted at spatial infinity, and it does not move the existential force outside the viscosity quantifier.

Status: Lean for the existential source-construction witness (the zero-mean-force theorem); this theorem preserves the source's noncomputable choices and its force formula.

```
-- Lean names mentioned in the force-mean note (paths relative to the book
--   FullMean.periodic_zero_mean_force                      (the zero-mean-
--   FullMean.selected_compact_mean_zero
--   ForceMean.candidate_force_mean_zero_before_one         (component calc
--   ConstructionMean.compact_force_mean_before_one         (vector integra
--   ConstructionMean.compact_borel_force_mean_zero         (all-time compa
```

Proof: proofs/OPENAI-COLUMN-FORCE-MEAN.md; Lean theorem named in the code block above.

Proposition 55: exclusions for OpenAI's construction at the matching viscosity

Credit: Builds on: OpenAI's construction (2026). Tools: Lean 4. This result (Sonny): proved (written).

Zero-mean and cell-energy-inequality competitors are excluded for periodic velocity at the matching viscosity; a fixed force has a periodic-pressure solution at large viscosity; a pressure-free reduction to forced Burgers (Lean equivalence); and there is a zero-force, zero-cell-mean pressure-free flow that is not periodic. These settle, for one force for every viscosity, zero mean velocity and cell energy inequality, the readings split as in section 4 of

chapter 8; the pressure-free readings are reduced. A cell energy inequality removes the pressure tilt (the mean-zero premise stated explicitly).

Status: written; the pressure-free reduction to forced Burgers is a Lean equivalence; the energy-mean algebra of the cell-energy-inequality proof is Lean (the PDE integration is not formalised).

Proof: proofs/OPENAI-COLUMN.md; proofs/OPENAI-COLUMN-ROWS.md.

Theorem 56: the fixed-viscosity one-cell-identity question is answered yes

Credit: This result (Sonny): proved (written).

One-cell identity on arbitrary vector tests forces matching traces and derivatives, so the fixed-viscosity case of the one-cell-identity question is excluded: for OpenAI's construction, with the erratum's label read literally (velocity periodicity dropped) and the corrected identity binding for periodic test fields on one cell, there is no competitor when the viscosity is fixed or chosen for each force, with no force convention other than "none", the divergence-free projection, or compact support in time (2,064,384 readings, written). Also: an exact pressure criterion and a periodic kinetic-source condition; finite monomial-periodic obstructions.

Status: written (a complete written proof). The literal-erratum question, the sublinear-pressure question and literal (B) remain open with reductions.

Proof: proofs/LITERAL-ERRATUM.md; proofs/LITERAL-ERRATUM-STATUS.md.

Proposition 57: the three qualifications of the one-cell identity

Credit: Tools: Lean 4. This result (Sonny): proved (written).

The zero-mean-force result removes one qualification of the one-cell identity: with the force required to have zero mean, the one-cell identity is excluded (688,128 readings, written, using the Lean zero-mean-force theorem); a fixed

force has a periodic solution at large viscosity, so with one force for all viscosities and the corrected identity binding in full it is answered: no (688,128 readings). The case with the pressure gradient required to vanish on the cell is open (688,128 readings).

Status: written, resting on the literal-erratum proofs, on the zero-mean-force result (Lean) and on the large-viscosity existence proof.

Proof: proofs/ONE-CELL-IDENTITY-QUALIFICATIONS.md.

Theorem 58: the ordinary printed (C) is answered yes for the checked port, with its variants

Credit: Builds on: OpenAI's construction (2026). Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (written).

Ordinary printed (C) is answered: yes for the checked port of the whole-space construction; positive-time smoothness, the whole-space energy inequality and a zero-integral force variant extend it; a displayed common-force family is impossible. In detail: (1) the usual energy inequality, force convention none or compact support in time: answered: yes (13,017,088 readings); (2) the energy bound kept and a norm of the pressure added: answered: yes (13,017,088 readings); (3) force convention "zero integral over all of space": answered: yes for the three readings of (7) that keep a bound (19,525,632 readings). Energy finite at each time only, and a divergence-free force, are open.

Status: written for the positive-time smoothness, zero-integral force and variants; Lean and nanoda for settlements 1 to 3 (nanoda accepted an export of 95,425 declarations containing them) and for the whole-space breakdown itself.

```
-- files lean/OrdinaryVariants.lean, ZeroIntegralForce.lean, EnergyAE.lean
--   CClosures.printed_C_energy_inequality      settlement 1
--   CClosures.printed_C_with_extra_condition   settlement 2
--   CClosures.printed_C_zero_integral_force    settlement 3
```

Proof: proofs/ORDINARY-PRINTED-C-VARIANTS.md; proofs/C-WEAK-CLASS-PROOFS.md; proofs/C-WRITTEN-SMOOTHNESS.md; proofs/C-WRITTEN-MEAN.md; proofs/C-FORMAL-CHECKS.md.

Theorem 59: (C) with weak solutions and the corrected identity tested through time zero

Credit: Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (written).

With the corrected identity tested through time zero and no initial term, a nonzero compactly supported start with zero force has no weak solution, in any weak class with a distributional initial value. So (C) with weak solutions, the erratum applied to all of space, zero-extended identity, is answered: yes in both weak classes (9,633,792 reading combinations of this kind).

Status: written, uses no cited theorem; also Lean and nanoda (an export of 95,425 declarations).

Proof: proofs/ZERO-EXTENDED-IDENTITY.md.

Theorem 60: OpenAI's construction, weak solutions, zero mean velocity

Credit: Builds on: OpenAI's construction (2026). Tools: Lean 4. This result (Sonny): proved (written).

For OpenAI's construction with weak solutions, pressure freedom, zero mean velocity with periodic velocity and repeating pressure: answered: no. The construction's force has zero mean, so a periodic Leray-Hopf solution from rest has zero mean velocity and competes with it (645,120 reading combinations).

Status: written; rests on a cited existence theorem (Leray 1934 and Hopf 1951, for periodic data) and on the zero-mean-force result (Lean). The logical combination alone is also checked in Lean.

```
--   NSRemaining.combine
--   CClosures.openai_weak_zero_mean_of_LH   (file lean/OpenAIWeakZeroMean
--     the Leray-Hopf step is the named hypothesis)
```

Proof: proofs/OPENAI-WEAK-ZERO-MEAN.md.

Theorem 61: (C) with one force and start for every viscosity: meaning 1 is answered no, by cited small-data theory; meaning 2 is open

Credit: Builds on: the classical small-data existence theorem (Fujita and Kato 1964, Kato 1972 and 1984, Giga and Miyakawa 1985; surveys Lemarie-Rieusset 2002 and Cannone 2004), cited and not formalised. Tools: Lean 4. This result (Sonny): proved (written).

Alternative (C) with one force for every viscosity has two meanings. Meaning 1, one start and one force: there are v_0 and f such that for every $\nu > 0$ no solution exists. This is false exactly when every pair (v_0, f) of the printed classes has a global smooth bounded-energy solution at some viscosity, which holds for large ν: with $u(x,t) = \nu w(x,\nu t)$, $p(x,t) = \nu^2 q(x,\nu t)$, $f(x,t) = \nu^2 F(x,\nu t)$ and $v_0 = \nu w_0$, the data of the viscosity-one problem shrink as ν grows, $\|w_0\|_{H^m} = \nu^{-1}\|v_0\|_{H^m}$ and $\int \|F(s)\|_{H^m}\, ds = \nu^{-1} \int \|f(t)\|_{H^m}\, dt$. Meaning 2, one force and a start chosen for each viscosity: open.

Status: answered: no for meaning 1 (written, with cited small-data theory; the scaling reduction is also Lean); meaning 2 open. The group of readings as a whole is open.

```
-- file: lean/CFScaling.lean
-- file: lean/CFFinal.lean
-- file: lean/ScalingInvariantForce.lean
```

Proof: proofs/C-COMMON-FORCE.md; proofs/C-READING-GATES.md; Lean files listed in the code block above (the scaling lemma: no nonzero force of class (5) is unchanged by a nontrivial rescaling of the equations).

Theorem 62: Clay's (A), (B) and (D) under the 2000 wording, stated directly

Credit: Builds on: OpenAI's construction (2026), which several of these concern or use. Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (Lean).

Stated directly against the comparator definitions of Clay's clauses: (B) answered: no and (D) answered: yes under energy on R^3, zero mean velocity, smooth zero extension, each pointwise degenerate identity, and each of the three binding identities with the text's test fields; (A) answered: no under smooth zero extension, the pointwise degenerate identities and the three binding identities, with the identity imposed on (A) itself. The (A) decision list of section 4 counts the identity readings of (A) with the erratum in the box only as answered: no where this result covers them (the text's test fields, smooth up to $t = 0$; 16,515,072 readings). Weak-solution forms in the trace weak class: Lean. The other test conventions: written.

Status: Lean; nanoda accepted exports of 93,579 declarations and, for the growth theorems, 91,888.

```
-- Clay's 2000 wording, stated directly (files lean/OWDefs.lean, OWA.lean,
--   OWIdent.lean, OWSuper.lean; build target ReadingsOriginalWording)
--   solution class and bridges: OriginalWording.ClaySolution, claySolutior
--     clayB_rest_iff, smoothSolution_of_clay; readings: ClayAWith, ClayBWi
--   (B): B_whole_space_energy_not_met, B_zero_mean_velocity_not_met, B_smo
--     B_stationary_not_met, B_pressure_cancels_not_met, B_curl_time_not_me
--     B_printed_identity_not_met_lean, B_time_corrected_identity_not_met_l
--     B_pressure_corrected_identity_not_met_lean
--   (A): A_smooth_zero_extension_not_met, A_stationary_not_met, A_pressure
--     A_curl_time_not_met, A_pressure_free_not_met, A_printed_identity_not
--     A_time_corrected_identity_not_met_lean, A_pressure_corrected_identit
--   (D): D_whole_space_energy_met, D_zero_mean_velocity_met, D_smooth_zero
--     D_stationary_met, D_pressure_cancels_met, D_curl_time_met, D_pressur
--     D_printed_identity_met, D_time_corrected_identity_met, D_pressure_co
```

Proof: chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 5, "Lean status of the written lemmas";

statement/STATEMENTS.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-statement-statements), collections of the second round.

Theorem 63: the additive shear, the strong form for OpenAI's U

Credit: Builds on: OpenAI's construction (2026). Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (Lean).

Let $\nu > 0$ and let (U, P) be OpenAI's periodic solution, as ported in this book, with force f. Let (u, p) solve the same equations with the same force for $0 < t < 1$, with u smooth there, $p(x + e_j, t) = p(x, t)$, and $u = U + H(t, \langle k, x \rangle)d$ with $\langle k, d \rangle = 0$, $d \neq 0$ and H any real function. Then for $0 < t < 1$, $U(x, t) = 0$ wherever $H(t, \langle k, x + e_j \rangle) \neq H(t, \langle k, x \rangle)$ for some j. The proof uses that the support of $U(\cdot, t)$ in the unit cell lies strictly inside it; for general periodic U the statement is false (Proposition 27).

Status: proved in Lean and nanoda (standard axioms only; nanoda accepted an export of 91,756 declarations containing them, log in certificates/additive-shear-nanoda.txt); written proof.

```
-- AdditiveShear.shear_forces_zero   (file lean/AdditiveShear.lean; build t
```

Proof: proofs/ADDITIVE-SHEAR.md; chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open).

Theorem 64: no additive-shear competitor for OpenAI's force

Credit: Builds on: OpenAI's construction (2026). Tools: Lean 4; the nanoda kernel checker. This result (Sonny): proved (Lean).

Fix $\nu > 0$, let f be the force of OpenAI's construction for this ν, the force behind the forced breakdown with repeating pressure, and let U be OpenAI's periodic solution. There is no pair (u, p) in the class of the literal-erratum question (C^∞ on $R^3 \times [0, \infty)$, equations (1) and (2) with this f, $u(\cdot, 0) = 0$, $p(x + e_j, t) = p(x, t)$, no periodicity or growth condition on u) with

$u(x,t) = U(x,t) + H(t,\langle k,x\rangle)d$ for $0 < t < 1$ and $\langle k,d\rangle = 0$, for any k, d and any real function H. It holds under weaker hypotheses: u smooth for $0 < t < 1$ and continuous on $R^3 \times [0,1]$, the same momentum equation for $0 < t < 1$, and p repeating for $0 < t < 1$. No growth bound, start from rest or divergence condition on u is used. The literal-erratum question itself stays open.

Status: proved in Lean and nanoda (standard axioms only; nanoda accepted an export of 91,756 declarations containing them, log in certificates/additive-shear-nanoda.txt); written proof.

```
-- in the literal-erratum class, with the force of the forced breakdown wit
--   AdditiveShear.chapter6_no_additive_shear
--   AdditiveShear.literal_no_additive_shear_competitor
-- weaker hypotheses, in the candidate's time-before-space order:
--   AdditiveShear.no_additive_shear_competitor
--   AdditiveShear.no_additive_shear_global_competitor
--   AdditiveShear.book_candidate_no_shear_competitor   (non-vacuous: the c
-- (file lean/AdditiveShear.lean; build target ReadingsAdditiveShear)
```

Proof: proofs/ADDITIVE-SHEAR.md; chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open).

Open questions

The classical (A) and (B)

Credit: Builds on: the questions of the Clay Mathematics Institute (C. L. Fefferman, 2000); Terence Tao. Localisation and compactness properties of the Navier-Stokes global regularity problem, arXiv:1108.1165 (2011), Analysis & PDE 6(1) (2013), 25-107.

(A): for $\nu > 0$, $n = 3$, u° any smooth divergence-free field satisfying (4) and $f \equiv 0$, do there exist smooth p, u_i on $R^3 \times [0,\infty)$ satisfying (1), (2), (3), (6), (7)? (B): the same for u° satisfying (8), with (10), (11) in place of (6), (7). Both stay open, under the 2000 wording and under the correction as intended. (D) for the printed wording at non-rest starts is open too (on every finite time interval, (D) holds exactly when (B) fails, by Tao's Proposition 1.7

(2011); for all time this holds only if the version his Remark 6.2 sketches, with the details omitted, holds; cited, not rechecked in Lean). Tao's unforced Conjecture 1.4 (Clay's (B) on the torus) stays open.

Status: open.

Proof: none; see chapter 5 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-05-other-answers-under-the-erratum) and chapter 8 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 6.

The literal-erratum question

Credit: Sonny, question stated exactly in chapter 6.

Fix $\nu > 0$ and let f be the force of OpenAI's construction for this ν, the force behind Corollary 10.6 and the forced breakdown with repeating pressure. Is there a pair $(u, p) \in C^\infty(R^3 \times [0, \infty))$ with $\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f$, $\mathrm{div} u = 0$, $u(\cdot, 0) = 0$, $p(x + e_j, t) = p(x, t)$ for $j = 1, 2, 3$, and no periodicity or growth condition on u? By the superlinear-growth theorem, any such u has $\sup_{R^3 \times [0, T]} |u|/(1 + |x|) = \infty$ for some $T < 1$; by the nonzero-defect lemma, any such u carries a nonzero period defect solving the pressure-free period-defect system from zero data. Yes would mean that, under reading (iii-a), OpenAI's force does not meet (D) from rest; no would mean that it does.

Status: open (research-level in both directions; eleven explicit constructions tried each fail for a stated reason, and no competitor of additive-shear form exists: Theorem 64).

Proof: none; see chapter 6 (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-06-what-remains-open).

The zero-mean-velocity and cell-energy-inequality questions

Credit: Sonny.

Is there a competitor for OpenAI's force with zero mean velocity for all t? Is there one that obeys the cell energy inequality and has a tilted pressure (periodic velocity), or a non-periodic velocity? For periodic velocity at the

matching viscosity the answer is no (3,440,640 readings). For one force at every viscosity it is yes (2,580,480 readings, so the construction is answered: no). With weak solutions and no $\nabla p = 0$ requirement it is also yes for zero mean velocity (645,120 readings). The remaining 3,268,608 readings, with free velocity or with $\nabla p = 0$, are open: the cell boundary flux is not controlled.

Status: open for the remaining 3,268,608 readings.

Proof: none; see chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 6.

The sublinear-pressure question

Credit: Sonny.

The literal-erratum class with "pressure sublinear in x" in place of "pressure periodic". An exact pressure primitive criterion and a finite monomial-periodic obstruction are written, but a general kinetic source periodicity is not established.

Status: open.

Proof: none; see proofs/LITERAL-ERRATUM-STATUS.md.

The one-cell-identity question with the pressure gradient required to vanish on the cell

Credit: Sonny.

For OpenAI's construction with the erratum's label read literally and the corrected identity imposed on spatially periodic tests on one cell: with one force for all viscosities and the pressure gradient required to vanish on the cell, 688,128 readings remain open.

Status: open.

Proof: none; see proofs/ONE-CELL-IDENTITY-QUALIFICATIONS.md.

One force for all viscosities, with the pressure gradient zero

Credit: Sonny.

For the ordinary periodic-pressure class it is answered: no (6,881,280 readings: a fixed force has a periodic-pressure solution at large viscosity). With $\nabla p = 0$ it is open (9,289,728 readings): excluding periodic forced Burgers competitors at every viscosity for one force.

Status: open for the $\nabla p = 0$ readings.

Proof: none; see chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 6.

Groups of readings decided in part

Credit: Sonny.

(A) and (C) with weak solutions and the erratum applied to all of space: 2,359,296 open readings for each (cell identities and pressure only at zero). OpenAI's construction with weak solutions and pressure freedom is open for 9,289,728 readings. For the ordinary printed (C), energy finite at each time only (19,525,632 readings) and a divergence-free force (26,034,176 readings) are open, each for one missing step stated in the printed (C) variants note. Also open: (C) with one force and a start chosen for each viscosity (meaning 2 of the common-force note).

Status: open.

Proof: none; see chapters/08-every-reading.md (https://gocheckyourself.org/read/navier-stokes/experts/#ch-chapters-08-every-reading), section 6, and proofs/ORDINARY-PRINTED-C-VARIANTS.md.

14 Check it yourself: Navier-Stokes,

the four alternatives and the erratum

Check it yourself: Navier-Stokes, the four alternatives and the erratum

This guide gives a written recipe for each claim in the book: the exact command, the exact condition to test, the answer to expect, and the standard public tool that checks it (Lean's build tool and axiom report, the nanoda kernel, or a few lines of code any reader can write). You can follow every step with public tools or code you write yourself. Each recipe also comes as a step-by-step page with commands for Windows, macOS and Linux and the exact output to expect. Lean names and commands are shown only in code blocks.

What is claimed

The question these claims answer: under each reading of Clay's 2000 wording of the Navier-Stokes problem and of the erratum, which of the four alternatives (A) to (D) is answered yes or answered no, or is still open, and how was each answer checked?

The formal checks support these statements. The exact Lean text of each is in the exact statements page, and the declaration names are in the certificate list.

Result	Statement, in plain words	Wording
The conditional zero-force drift	With no force and the fluid at rest, the printed periodic class allows both the resting solution and a uniformly sliding one.	printed
The bounded drift	The same, for an explicit slide with speed below 1 and bounded energy in each cell.	printed
The slide-through escape	For every viscosity and every admissible force (smooth, periodic, decaying as in (8), (9)), there is a smooth solution from rest that lasts forever, with periodic velocity and a pressure that tilts across space.	printed
The escape-pressure criterion	That escape solution's pressure is periodic exactly when the slide's acceleration equals the force's spatial mean at every time.	both
The failure of the escape under repeating pressure	Adding periodic pressure makes the universal escape statement of the slide-through escape false.	erratum
The forced breakdown with repeating pressure	For every viscosity there is an admissible force for which no smooth solution from rest, periodic in velocity and in pressure, lasts forever.	erratum
The three wider-class statements	The same conclusion, stated as the failure of universal existence over broader force classes (every smooth repeating force; the Formal Conjectures repeating force class with a zero start; and every repeating starting flow with that class, in the Formal Conjectures comparator vocabulary).	erratum
The whole-space breakdown	(C), all of space: for every viscosity there is a decaying starting flow and a decaying force for which no smooth solution with bounded energy exists for all time (Formal Conjectures comparator quantifiers).	printed; the erratum names equations (8) and (10) only, which (C) does not cite
The equality of the two force classes	The force class of the forced breakdown with repeating pressure and the force class of the slide-through escape are the same.	both
The equivalence of printed and periodic-pressure existence for (B)	For unforced (B) and every starting flow, a smooth solution with periodic velocity exists exactly when one with periodic velocity and periodic pressure exists.	both

The forced breakdown with repeating pressure and its three wider-class statements rest on OpenAI's construction (NavierStokesAndEuler, Apache-2.0). The comparison of each statement with Clay's printed clauses and the erratum is in the answer key and the chapters.

Status

- Lean-checked (Lean 4.34.1, Mathlib d13f23b7, standard axioms only): the ten results above and the two bridges.
- Files. The Lean folder holds 903 files, of which 898 are Lean files (these include OpenAI's Apache-2.0 package, with 70 whole-space files and one overlay added for (C)); the other 5 are the build file, the lockfile, the toolchain file, the licence and the notice.

- Also accepted by separately written kernels:
  - The slide-through escape and the forced breakdown with repeating pressure: nanoda 0.4.26, in the two-root export of 54,859 declarations, and again as roots of the eleven-root export below. lean4lean 67f63b0c replayed the whole environment of the default target, which contains both (816,915 declarations, every imported Lean and Mathlib declaration included), into a fresh kernel, with no errors (certificates/kernel-checks.txt).
  - The three wider-class statements: nanoda 0.4.26, in an export of 91,636 declarations ("Checked 91636 declarations with no errors").
  - The whole-space breakdown (alternative (C), the whole-space comparator theorem): nanoda, in an export of 94,908 declarations; with the shipped build file, the added build target completed.
  - The additive shear (Theorems 63 and 64, file lean/AdditiveShear.lean, build target ReadingsAdditiveShear): nanoda, in an export of 91,756 declarations ("Checked 91756 declarations with no errors", certificates/additive-shear-nanoda.txt).
  - Rerun on the shipped Lean folder after the last file renames and comment edits: all seven build targets completed from the shipped build file, one target at a time, with no errors and no unproved-placeholder warnings. The axiom lines printed during those builds (872 declarations) list only propext, Classical.choice and Quot.sound, or fewer (certificates/axiom-printout.txt); the printout for the answer key is in certificates/answer-key-axioms.txt. nanoda accepted the eleven-root export (94,342

declarations) and the whole-space export (94,908 declarations) with no errors (certificates/kernel-checks.txt). lean4lean replayed the whole environment of the default target (816,915 declarations) with no errors; it contains the eleven roots of the eleven-root export, that is, every result in the table except the whole-space breakdown.
    - The two drift results, the escape-pressure criterion and the failure of the escape under repeating pressure: Lean, and nanoda (each is a root of the eleven-root export), and lean4lean (each lies in the environment it replayed).

- Erratum reading. The forced breakdown with repeating pressure and its three wider-class statements are Lean-checked, under the erratum reading: keep velocity periodicity (10), correct (12), require periodic pressure of solutions. The literal label "Eqn (10) should read" gives a larger class of competitors. Chapter 6 treats it: answered: no without periodic pressure (the slide-through escape); open with periodic pressure, narrowed to competitors that grow faster than linearly. Chapter 8 covers every other reading and gives the decision lists from which its counts can be recomputed (recipe below).
- Bridges. The two bridges are Lean-checked in the default build target. Each is a root of the eleven-root export (nanoda) and lies in the environment that lean4lean replayed. They are not in the nine lines of the axiom printout for the answer key; they are listed in the certificate list.

- Scope. Chapter 8 gives a verdict for each of (A) to (D) under 440,401,920 readings of the text. Its basis is 12 Lean-checked statements, five collections of Lean helper statements of the first round (310 declarations) and seven of the second round, 15 lemmas (two in Lean in full, the pressure-periodicity lemma and the curl of OpenAI's force; 13 written proofs with parts in Lean, two of them in Lean only conditional on a named premise), written proofs in the proofs folder (start with the closure proofs) and cited classics, and every verdict is

labelled. The count includes every split group of readings, the closing of (C) and the count of (A) under the identity readings.

- The logic-only file. The small logic-only file in the Lean folder (named in the code block below) was compiled with Lean 4.34.1; it printed that the declaration does not depend on any axioms. It proves only the combination of two results that it takes as hypotheses.

- The added Lean target. The added build target builds the first-round Lean sources (the two lemma collections, the literal-erratum lemmas, the close-out modules, the force-mean modules, the local-research file) and the whole-space closure of (C) (70 files added to the book's Lean folder for the construction). With the build file as shipped, it completed (Lean 4.34.1, 9,657 build steps), on top of a from-source build of the default target with the pinned Mathlib reused. A build from an entirely clean checkout, with nothing reused, was not run. For (C), an overlay of the comparator definitions is the file used in the checked build (explicit imports instead of the Mathlib umbrella import, a modification notice added; declarations unchanged); the book's own copy of the comparator definitions is the upstream bytes. Twenty-one other files carry marked compatibility overlays (the port notes): four were already in the book in their effective form (listed in the notice file) and 17 are among the added files, shipped in their effective form.

- The closure list for (C). For every row of the closure list whose in_release field is true, the file in the Lean folder must match original_sha256 when release_identical_to_original is true, and effective_sha256 when it is false. The one file with an overlay, the comparator definitions, ships as the original; its overlay is in the overlays folder of the Lean folder and matches effective_sha256. From the bundle root, the python command prints 522 522 (the rows, then the matches), and the sha256sum command prints the overlay's hash, which begins 39bda0f1 and equals effective_sha256 in that row:

```
python -I -c "import csv,hashlib;R=[r for r in csv.DictReader(open('cert:
sha256sum lean/overlays/NavierStokes/ComparatorDefinitions.lean
```

- The second-round Lean files. Clay's 2000 wording stated directly and the two growth theorems of chapter 6 (in a third build target), the whole-space closures of (C) (added to the first-round target), and weak solutions (in a fourth build target). In the second round all three targets completed from the shipped build file, on top of the earlier build (Lean 4.34.1, 9,690 build steps). Every axiom line printed during the build lists only propext, Classical.choice and Quot.sound. nanoda accepted four exports containing the new theorems: 93,579 declarations (the 2000 wording), 91,888 (the growth theorems), 95,425 (the closures of (C)) and 50,100 (weak solutions) (earlier measurement, not re-measured). The one-push files (a fifth build target, for the readings with one force for every viscosity) were compiled one file at a time, and the target was later built from the shipped build file; every axiom line lists only the same three axioms, and nanoda accepted an export of 53,132 declarations (earlier measurement, not re-measured) containing their final theorems. The scaling lemma (a root of the first-round target) was compiled the same way, and nanoda accepted an export of 33,099 declarations (earlier measurement, not re-measured).

- Tao's Conjecture 1.8 and the written-rows part. Two further build targets, built from the shipped build file (earlier, one file at a time). One proves that OpenAI's periodic corollary implies that Tao's Conjecture 1.8 (arXiv:1108.1165, p. 5) is false as Tao wrote it, under each of three readings of his normalised pressure and for a solution of any spatial period (chapter 4); nanoda accepted exports of 91,179 declarations (period 1) and 95,687 declarations (any period; the export contains the period-1 theorem) (earlier measurement, not re-measured). The other formalises the smooth-solutions identity, the curl of OpenAI's force, the start-level readings and the pressure-on-all-of-space lemma for solutions smooth up to $t = 0$ (chapter 8, section 5); nanoda accepted an export of

94,345 declarations (earlier measurement, not re-measured). Every axiom line lists only the same three axioms.

What was rebuilt from source and what was not

The default build target was built from source, with the pinned Mathlib reused from a pre-built copy. The added targets of the first round and three of the second-round targets (Clay's 2000 wording with the two growth theorems, the whole-space closures of (C), and weak solutions) completed from the shipped build file on top of that build. After the last renames, all seven build targets (including the one-push files, the Conjecture 1.8 files and the written-rows part of chapter 8) were built from the shipped build file, one target at a time. In every build, the compiled objects of the inherited Apache-2.0 package come from copies and were not rebuilt. In the second round the Lean builds, the axiom printouts and the three nanoda exports (54,859, 91,636 and 94,908 declarations) were rerun on a byte-checked copy of an existing build, whose Lean files are identical to this book's. After the last renames, nanoda was run on the shipped Lean folder on two exports: the eleven-root export (94,342 declarations) and the whole-space export (94,908) (certificates/kernel-checks.txt). Each export has its own roots and so its own number of declarations; the numbers in this guide belong to different exports, and each is listed with its roots in the certificate list. No build from a clean checkout was run.

Commands

From the bundle root, with elan installed (the Lean folder pins the toolchain and the library versions):

```
cd lean
lake exe cache get
lake build
```

The axiom printout for the answer key. Put the nine lines in a file inside the Lean folder that begins with the import of the default target, and run it with the build tool's environment:

```
import ReadingsCore

-- Each line should list only propext, Classical.choice and Quot.sound.
#print axioms ClayWording.zero_data_nonuniqueness_in_printed_problem
#print axioms ClayWording.bounded_energy_nonuniqueness_in_printed_problem
#print axioms JustCheckin.A3.escape_from_rest
#print axioms JustCheckin.A3.PressureBoundary.escape_pressure_periodic_iff
#print axioms JustCheckin.BoundaryMap.periodic_escape_refuted
#print axioms ForcedBreakdown.forced_periodic_from_rest
#print axioms JustCheckin.BoundaryMap.periodic_smooth_existence_refuted
#print axioms JustCheckin.BoundaryMap.periodic_decay_existence_refuted
#print axioms JustCheckin.BoundaryMap.openai_D_universal_existence_refuted
```

```
lake env lean <that file>
```

The first-round sources (not part of the default target; this needs hours, and keep parallel compiles low):

```
lake build ReadingsClose
lake env lean NSContinuation/AxiomRun.lean
lake env lean NSContinuation/LiteralAxioms.lean
lake env lean AxiomAudit.lean
```

The last three files print axiom lists. The file for the second collection prints 204 of them; the other two print the axioms of the four literal-erratum lemmas and of the force-mean modules. To print the axioms of the whole-space breakdown and six supporting results, put these lines in a file that begins with the import of the whole-space comparator theorem and run it the same way:

```
import NavierStokes.ComparatorR3Theorem
#print axioms NavierStokes.ComparatorBridge.navier_stokes_breakdown_R3
#print axioms NavierStokesR3.theorem_1_1
#print axioms NavierStokesR3.theorem_1_1_with_initial_rest
#print axioms NavierStokesR3.WholeSpaceUniqueness.classical_uniqueness_on_
#print axioms NavierStokesR3.PressureRecovery.pressure_gradient_recovery
#print axioms NavierStokesR3.PressureFlux.exists_uniform_actual_pressure_f
#print axioms NavierStokesR3.ProblemStatement.zero_force_has_global_solutio
```

The one small Lean file added in the evening checks only a logical step, with two earlier results as hypotheses. It imports nothing and is in no build target. From the Lean folder, run it with the pinned toolchain:

```
lean Combine.lean
# expected last line: 'NSRemaining.combine' does not depend on any axioms
```

The second-round files. Each file of the 2000-wording, weak-solution and one-push targets, and the scaling lemma, prints the axioms of its theorems while it builds. The two audit files print the axioms of the closures of (C):

```
lake build ReadingsClose ReadingsOriginalWording ReadingsWeakClass Readings
lake env lean CClosuresAuditA.lean
lake env lean CClosuresAuditB.lean
# every printed axiom list: [propext, Classical.choice, Quot.sound]
```

Tao's Conjecture 1.8 (chapter 4) and the written-rows part of chapter 8, two further targets. Each file prints the axioms of its final theorems while it builds:

```
lake build ReadingsTao18 ReadingsWrittenRows
lake build ReadingsAdditiveShear   # the additive shear of chapter 6
# every printed axiom list: [propext, Classical.choice, Quot.sound]
# nanoda: 91,179 and 95,687 declarations (Tao 1.8), 94,345 (written rows) (
```

Separate kernel check with nanoda. Build lean4export 3.1.0 against the same toolchain and put its build output on the Lean library path (LEAN_PATH) beside the project's. Then, inside the Lean folder:

```
lake env lean -M13000 -DmaxRecDepth=50000 -DmaxHeartbeats=0 --run <lean4exp
  JustCheckin.BoundaryMap.periodic_smooth_existence_refuted \
  JustCheckin.BoundaryMap.periodic_decay_existence_refuted \
  JustCheckin.BoundaryMap.openai_D_universal_existence_refuted > export.nd
nanoda_bin config.json
```

where config.json is nanoda's standard configuration, permitting only the three standard axioms:

```
{
  "export_file_path": "export.ndjson",
  "permitted_axioms": ["propext", "Classical.choice", "Quot.sound"],
  "unpermitted_axiom_hard_error": true,
  "nat_extension": true,
  "string_extension": true,
  "print_success_message": true,
  "print_axioms": true,
  "num_threads": 1,
  "pp_to_stdout": true,
  "pp_declars": []
}
```

To check the slide-through escape and the forced breakdown with repeating pressure the same way, export from the module of the default target with these two roots:

```
JustCheckin.A3.escape_from_rest
ForcedBreakdown.forced_periodic_from_rest
```

Recount of the reading combinations

The file format is the decision lists in chapter 8 (section 4) and the sub-reading descriptions in the reading splits. The condition to test: enumerate the 3,440,640 combinations of the 11 clauses of chapter 8 that are not set

aside in its section 3 (each clause has the values listed there), give each combination to the decision list of each of the five columns, where the earliest matching row decides, and confirm that every combination is decided exactly once. Multiply every count by 128, the number of combinations of the five clauses set aside. The factor 128 cannot reproduce the split of the ordinary printed (C) rows by condition (7): to recount those rows, enumerate condition (7) as well (12 clauses, 13,762,560 combinations) and multiply every count by 32. Expected answer: a total of 440,401,920 combinations per column and the shares in the table of the README; the number of distinct patterns of verdicts across the five columns is 82 (one pattern per combination, listing the five verdicts; a verdict that depends on condition (7) enters as the set of its verdicts). A few lines in any language are enough.

Expected output

- The default build finishes with no errors and no unproved-placeholder warnings (9,636 build steps).
- Each of the nine lines of the axiom printout lists only the axioms propext, Classical.choice and Quot.sound. It was run against a completed build of sources byte-identical to the Lean folder (9,636 build steps), and all nine lines listed exactly these three axioms.
- The added build target finishes with no errors; every axiom line printed by the files above lists only the same three axioms.
- nanoda prints "Checked N declarations with no errors" and lists only propext, Quot.sound and Classical.choice. N depends on the roots chosen; for the three roots of the wider-class statements it was 91,636.
- The recount gives 440,401,920 combinations per column and the shares in the README.

How I direct this work, and my own tools, are not part of this book. Everything you need to check the results is public: the results, their certificates and fingerprints, the Lean proofs, and a written recipe for each check.

Supporting records are available on request.

Time and memory

- The full default build runs 9,636 steps, including 712 project modules. We did not time it in one uninterrupted sitting; on a desktop machine, expect hours rather than minutes.
- Memory: keep parallel compiles low (we used four threads; 14 parallel compiles used about 15 GB). On Windows, build from a short folder path.
- Export of the three wider-class statements: about 2.5 minutes, peak about 10 GB, and it needs the memory flag shown above. nanoda on that export: about 4 minutes on one thread.
- The recount: seconds to minutes, depending on the language.

What is not claimed

- No claim about the Clay prize. The Clay Mathematics Institute decides, by its official problem description and at its sole discretion, whether a proposed solution answers a Millennium Problem.
- No claim about which wording governs, or about the publisher's intent behind the erratum.
- No claim that the Lean checks confirm OpenAI's written paper. The checks concern the Lean statements only; whether each step of the paper matches the Lean proof is a separate question (chapter 7).

- No answer to (A) or (B) under the text's own wording with nothing added (the classical (A) question and the classical (B) question of chapter 8), none for printed (D) from starts other than rest (the remaining readings of (D)). Under the other readings, chapter 8 gives a labelled verdict for each alternative. Several groups keep open cases (listed in chapter 8, section 4 and the reading splits), among them the full families of the ordinary printed (C) and its variants and of (C) with weak solutions and the erratum applied to all of space.
- No settled claim about the literal "Eqn (10)" reading with periodic pressure (the literal-erratum question). It is open, and chapter 6 states it exactly. The fixed-viscosity one-cell-identity question is excluded by a written proof, not by Lean.

- Not rebuilt here: the compiled objects of the inherited Apache-2.0 package, which the checked builds take from copies.

- The mathematics of the construction is OpenAI's; it is accepted by the Lean kernel and by the separately written checkers named above, not by a human referee here.

Limits

Each limit of the checks is listed once, here.

- Review. No person outside this project has read these checks or the written proofs, and none of the results is refereed.
- lean4lean. lean4lean replayed the whole environment of the default target, which contains every result in the table except the whole-space breakdown; it replayed an environment, not an export. The whole-space breakdown and the other build targets were checked by Lean and, where stated, nanoda. The replay ran in a build copy of the shipped Lean folder whose Lean files are the same as the shipped ones.
- Builds. All seven build targets were built from the shipped build file after the last renames (the pinned Mathlib and the inherited package's compiled objects reused from a pre-built copy). A build from an entirely clean checkout, with nothing reused, was not run.

- Lean reuse. The Lean files reuse OpenAI's Lean library as a base: about 383,000 lines in 609 files (the 539 files used for the periodic result and the 70 files of its whole-space part, added for (C)), with about 24,700 theorems and lemmas (counted in the ported copy). Rewriting it would take one professional at least a year and a half of full-time work, even at an unusually fast 1,000 finished lines a day, and only certain theorems needed testing. The statements and proofs that test the other parts of the problem are Sonny's.
- Written steps. Steps not in Lean are labelled written where they occur. OpenAI's Corollary 10.6 is used as their Lean theorem states it.

Where to go next

- Which verdicts rest on Lean and which only on written proofs? See Every reading.
- What is still open after the Lean checks? See What remains open.

We hope it helps others find more results and keeps their curiosity going. We'd love to collaborate with anyone.

15 Files and fingerprints

This chapter is the book's file SHA256SUMS.txt.

16 Trust and verification

This chapter is the book's file VERIFY.md.

17 Lean sources: how to check them

This chapter is the book's file THEOREMS.md.

18 People and works

This chapter is the book's file CREDITS.md.

19 Files and rights

Licence

The files Sonny wrote for this book are under the Creative Commons Attribution 4.0 International licence (CC BY 4.0). You may copy, share and adapt them for any purpose, as long as you give credit.

How to give credit: "Sonny (Sonny), gocheckyourself.org", with a link to the page or file you used, and a note saying what you changed, if anything.

What it covers: the chapters, the answer key, the statement pages, the written proofs, the certificates and data files, and the Lean files outside the construction folder, except the standalone copy of the Formal Conjectures statement.

What it does not cover:

- Every file from OpenAI's NavierStokesAndEuler, and the Formal Conjectures comparator files (the comparator definitions and the modified standalone copy of the Formal Conjectures statement). They stay under the Apache License, Version 2.0. Their notices are kept, and the licence and the notice file ship with them.
- Programs that are not part of this book, and work that is not published on this site.
- Trade marks, and the other rights that section 2(b) of the licence leaves out.

The licence file is LICENSE-CC-BY-4.0.txt; the full legal code is at https://creativecommons.org/licenses/by/4.0/legalcode.

Where to go next

- Whose work does this book build on? See the credits page.
- How do you rebuild and recheck the Lean files? See the check guide.

20 Articles

- The Navier–Stokes scorecard: what each wording of Clay's problem answers
- The Pressure Loophole: How the Wording Decides the Answer

21 Each result on its own page

1. Theorem 1: the conditional zero-force drift STATUS: LEAN-CHECKED (SECOND-KERNEL)

2. Theorem 2: the bounded drift STATUS: LEAN-CHECKED (SECOND-KERNEL)
3. Theorem 3: the slide-through escape STATUS: PROVED IN WRITING (PARTLY-LEAN)
4. Theorem 4: the escape-pressure criterion STATUS: LEAN-CHECKED (SECOND-KERNEL)
5. Theorem 5: the failure of the escape under repeating pressure STATUS: LEAN-CHECKED (SECOND-KERNEL)
6. Theorem 6: the forced breakdown with repeating pressure STATUS: LEAN-CHECKED (SECOND-KERNEL)
7. Theorem 7: the two force classes are the same STATUS: LEAN-CHECKED (SECOND-KERNEL)
8. Theorem 8: with no force, (B) under the printed pressure and under periodic pressure are the same question STATUS: LEAN-CHECKED (SECOND-KERNEL)
9. Theorem 9: the breakdown for every smooth repeating force STATUS: LEAN-CHECKED (SECOND-KERNEL)
10. Theorem 10: the breakdown in the Formal Conjectures force class STATUS: LEAN-CHECKED (SECOND-KERNEL)
11. Theorem 11: the breakdown for every repeating starting flow STATUS: LEAN-CHECKED (SECOND-KERNEL)
12. Theorem 12: the whole-space breakdown STATUS: LEAN-CHECKED (SECOND-KERNEL)
13. Theorem 13: Corollary 10.6, the periodic breakdown STATUS: CITED
14. Proposition 14: Tao's Proposition 1.7, elimination of the forcing term STATUS: CITED
15. Lemma 15: the pressure of a smooth periodic solution STATUS: CITED
16. Theorem 16: Tao's Conjecture 1.8 is false as Tao wrote it STATUS: LEAN-CHECKED (SECOND-KERNEL)
17. Lemma 17: smoothness on the closed slab STATUS: PROVED IN WRITING

18. Lemma 18: Tao's pressure meets each reading STATUS: PROVED IN WRITING
19. Lemma 19: the shift lemma STATUS: LEAN-CHECKED (SECOND-KERNEL)
20. Lemma 20: the period-defect system STATUS: LEAN-CHECKED (SECOND-KERNEL)
21. Lemma 21: the nonzero-defect lemma STATUS: LEAN-CHECKED (SECOND-KERNEL)
22. Lemma 22: the subclass-exclusion lemma STATUS: LEAN-CHECKED
23. Theorem 23: at most linear growth forces periodicity, the linear-growth repetition theorem STATUS: LEAN-CHECKED (SECOND-KERNEL)
24. Theorem 24: for OpenAI's force, growth must be superlinear before time 1, the superlinear-growth theorem STATUS: LEAN-CHECKED (SECOND-KERNEL)
25. Proposition 25: the class (iii-a) has fast-growing members STATUS: PROVED IN WRITING
26. Proposition 26: uniqueness already fails on a time slab for OpenAI's force STATUS: PROVED IN WRITING (PARTLY-LEAN)
27. Proposition 27: the additive shear, a counterexample to the strong form STATUS: PROVED IN WRITING
28. Proposition 28: the additive shear, the weaker statement that holds for every periodic U STATUS: PROVED IN WRITING
29. Computation 29: the erratum's sign pattern and the defect identities, computed STATUS: COMPUTED; THE PROGRAM IS NOT PUBLISHED HERE
30. Lemma 30: the smooth-solutions identity STATUS: PROVED IN WRITING (PARTLY-LEAN)
31. Lemma 31: the extra-identity reduction STATUS: PROVED IN WRITING (PARTLY-LEAN)
32. Lemma 32: the curl of OpenAI's force STATUS: PROVED IN WRITING (PARTLY-LEAN)

33. Lemma 33: the frozen-vorticity obstruction STATUS: PROVED IN WRITING (PARTLY-LEAN)
34. Lemma 34: the shear-start obstruction STATUS: PROVED IN WRITING (PARTLY-LEAN)
35. Lemma 35: the witness forces for (D) STATUS: PROVED IN WRITING (PARTLY-LEAN)
36. Lemma 36: the whole-space energy exclusion STATUS: PROVED IN WRITING (PARTLY-LEAN)
37. Lemma 37: tilt removal while the force is zero STATUS: PROVED IN WRITING (PARTLY-LEAN)
38. Lemma 38: the positive-time exclusion STATUS: PROVED IN WRITING (PARTLY-LEAN)
39. Lemma 39: the coarser-period reduction STATUS: PROVED IN WRITING (PARTLY-LEAN)
40. Lemma 40: the pressure-periodicity lemma STATUS: LEAN-CHECKED
41. Lemma 41: the cell-test detection lemma STATUS: PROVED IN WRITING (PARTLY-LEAN)
42. Lemma 42: the divergence-free force reduction STATUS: PROVED IN WRITING
43. Lemma 43: the equivalence lemma for (B) STATUS: PROVED IN WRITING (PARTLY-LEAN)
44. Lemma 44: the pressure-on-all-of-space lemma STATUS: LEAN-CHECKED (SECOND-KERNEL)
45. Theorem 45: Seeley's extension theorem STATUS: CITED
46. Theorem 46: global weak solutions STATUS: CITED
47. Proposition 47: the identity algebra of the trace weak class STATUS: PROVED IN WRITING (PARTLY-LEAN)
48. Theorem 48: one primitive periodic start settles the weak degenerate-identity groups of (D) and (B) STATUS: PROVED IN WRITING (PARTLY-LEAN)

49. Theorem 49: a decaying start and the periodic-pressure obstruction STATUS: PROVED IN WRITING (PARTLY-LEAN)
50. Theorem 50: the pressure at time zero only, with smoothness only afterwards, for (A) STATUS: PROVED IN WRITING
51. Proposition 51: weak pressure at zero and one-cell identities STATUS: PROVED IN WRITING
52. Proposition 52: OpenAI's construction with weak solutions and pressure freedom STATUS: PROVED IN WRITING
53. Proposition 53: the minimal-period (D) group, primitive starts STATUS: PROVED IN WRITING
54. Theorem 54: the construction's force has exactly zero mean at every time STATUS: PROVED IN WRITING (PARTLY-LEAN)
55. Proposition 55: exclusions for OpenAI's construction at the matching viscosity STATUS: PROVED IN WRITING
56. Theorem 56: the fixed-viscosity one-cell-identity question is answered yes STATUS: PROVED IN WRITING
57. Proposition 57: the three qualifications of the one-cell identity STATUS: PROVED IN WRITING (PARTLY-LEAN)
58. Theorem 58: the ordinary printed (C) is answered yes for the checked port, with its variants STATUS: PROVED IN WRITING (PARTLY-LEAN)
59. Theorem 59: (C) with weak solutions and the corrected identity tested through time zero STATUS: PROVED IN WRITING (PARTLY-LEAN)
60. Theorem 60: OpenAI's construction, weak solutions, zero mean velocity STATUS: PROVED IN WRITING (PARTLY-LEAN)
61. Theorem 61: (C) with one force and start for every viscosity: meaning 1 is answered no, by cited small-data theory; meaning 2 is open STATUS: PROVED IN WRITING (PARTLY-LEAN)
62. Theorem 62: Clay's (A), (B) and (D) under the 2000 wording, stated directly STATUS: LEAN-CHECKED (SECOND-KERNEL)

63. Theorem 63: the additive shear, the strong form for OpenAI's U

    STATUS: LEAN-CHECKED (SECOND-KERNEL)

64. Theorem 64: no additive-shear competitor for OpenAI's force STATUS: LEAN-CHECKED (SECOND-KERNEL)

65. Open question: The classical (A) and (B) STATUS: OPEN

66. Open question: The literal-erratum question STATUS: OPEN

67. Open question: The zero-mean-velocity and cell-energy-inequality questions STATUS: OPEN

68. Open question: The sublinear-pressure question STATUS: OPEN

69. Open question: The one-cell-identity question with the pressure gradient required to vanish on the cell STATUS: OPEN

70. Open question: One force for all viscosities, with the pressure gradient zero STATUS: OPEN

71. Open question: Groups of readings decided in part STATUS: OPEN

© 2026 Sonny. The books are under CC BY 4.0 (Creative Commons Attribution 4.0 International): reuse them with credit and a link. Each book's licence file lists the files that keep their own licences.