Erdős problems 64 and 647

READING LEVEL EVERYONE · EXPERTS

Computed: no counterexample to problem 64 on up to 22 vertices (credit: Ducoffe and Dumitru (every graph to 40 vertices); Garcia (to 23, checkable proofs); graphs generated with nauty's geng (McKay and Piperno); an independent replication to 22 vertices by Sonny), and no example for problem 647 up to 10^{14}; both problems open; not yet refereed.

proved in writing 5 · computed; the program is not published here 4 · computed 4 · proved in writing; the program is not published here 2 · open 2 · cited 1

Overview

Paul Erdős and András Gyárfás asked whether every finite graph in which each dot has at least three lines must contain a closed loop whose length is a power of two, such as 4, 8 or 16. A graph with no such loop would be a counterexample. Hong Liu and Richard Montgomery showed the answer is yes once the minimum degree is large enough, so the open part is sparse graphs. This book reports a computer search that I ran. Three reductions proved in the chapter narrow the search to one class of graphs, and an exhaustive search of that class at every number of dots from 4 to 22 found nothing. So every graph with minimum degree at least 3 on at most 22 vertices has a cycle of length 4 or 8. The status is computed, not yet refereed. A separate program of ours repeated the search up to 15 vertices, and preprints report larger ranges, up to 23 and 40 vertices. Orders 16 to 22 used a pruned build of a standard graph generator that is not published, so the shipped logs show that every part finished and found nothing. Order 23 and above are not covered here, and the problem stays open. A reader can check the logs with the written recipes and the SHA-256 fingerprints; the recipes check bookkeeping and do not rerun the searches.

CHECK IT YOURSELF

The check guide for this book · How to check a result yourself

Cite: Sonny (2026). Erdős problems 64 and 647 (preliminary). Go check yourself. https://gocheckyourself.org/read/erdos-problems/everyone/

1. Start here
2. Erdős #64: cycles of power-of-two length
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4. Try it yourself

Start here

Sonny

In short. Two open questions that Paul Erdős posed, each checked by an exhaustive computer search. For problem 64, no graph (dots, called vertices, joined by lines) with minimum degree (the fewest lines at any vertex) at least 3 on up to 22 vertices is a counterexample (a case that would show the claim is false); for problem 647, there is no example up to 10^{14}, and both problems are still open.

What this book shows

- My result, problem 64. No graph with minimum degree at least 3 on up to 22 vertices lacks a cycle of length 4 or 8. Computed: three reductions proved in the chapter narrow the search to one class of graphs, and an exhaustive search of that class at every order (number of vertices) from 4 to 22 found none, so the result covers all graphs with minimum degree at least 3, not just cubic (every vertex has exactly 3 neighbours), bipartite (the vertices split into two groups and every line joins the two groups) or small-diameter (any two vertices are a few steps apart) ones. A separate program of ours repeated the search up to 15 vertices. Orders 16 to 22 were searched with a pruned build of nauty's geng (a standard program that lists graphs of a chosen kind) that is not published; its logs are. Two recent preprints (papers shared publicly before peer review) report larger ranges (23 and 40 vertices).
- My replication, problem 647. No n with $24 < n \le 10^{14}$ satisfies the condition. Computed: up to 10^8 by two separately written programs, up to 10^9 also by a separate scan, and from 10^9 to 10^{14} by one sieve program (a program that strikes out numbers in bulk) that is not published; its logs are, and the check guide checks their bookkeeping. This replicates a smaller range than the forum reports.

- Why it matters. For problem 64 the open part is graphs of small minimum degree, such as three, and my search covers all of them up to 22 vertices. For problem 647, any example would be very large. Both problems are open.
- In Lean: nothing (Lean is a proof-checking program that tests every step of a proof against fixed rules). The checks are the search logs, the separate programs and a written recipe for each check.
- Every result in this book is listed in the theorem library.

The question

This book is about two open questions that Paul Erdős posed, collected with their history on Thomas Bloom's site erdosproblems.com. The wording below is our plain-words paraphrase; the exact wording is on each problem page.

Problem 64 (erdosproblems.com/64). Take any finite graph in which every vertex has at least three neighbours. Must it contain a cycle (a closed loop of distinct vertices) whose length is a power of two: 4, 8, 16, 32 and so on? Erdős and András Gyárfás conjectured the answer; the problem page says they "believed the answer must be negative", and that for every r there must be a graph of minimum degree at least r with no such cycle. The page lists a prize of $1000 and does not say who offered it.

Problem 647 (erdosproblems.com/647). Write $\tau(m)$ for the number of divisors of m. Is there an integer n above 24 such that $m + \tau(m)$ is at most $n + 2$ for every m below n? Erdős and John Selfridge asked this, and Erdős offered 25 pounds for an example. The numbers 1, 2, 3, 4, 5, 6, 8, 10, 12 and 24 satisfy the condition.

Where to go next. Why does a search up to 22 vertices leave problem 64 open? See the chapter on problem 64.

Why it matters

The problem pages give the questions and the prizes; we have not found a statement of Erdős's reasons and we do not guess them.

- Problem 64 asks how much room a sparse graph has to avoid every cycle length that is a power of two. Hong Liu and Richard Montgomery showed the answer is yes once the minimum degree is larger than some absolute constant, so the open part is graphs with small minimum degree, such as three.
- Problem 647 asks whether the divisor function can stay small enough, just below an integer, for all smaller numbers at once. Forum posts report that searches have ruled out every n up to 10^{22}, with a certificate (a file that lets a reader check the claim) only up to about 6×10^{17}, so any example would be very large.

Where to go next. What did the earlier searches for problem 64 cover, and what did they leave out? See the chapter on problem 64.

What answering it would help with

- A graph with no power-of-two cycle (problem 64) would show that the large-minimum-degree theorem needs its degree condition; a proof for every graph would determine which cycle lengths minimum degree forces.
- An example, or a proof that none exists (problem 647), would determine whether the divisor function can stay within the stated limits on consecutive integers.

Where to go next. What would a counterexample to problem 64 have to look like? See the chapter on problem 64. How large are the numbers the forum has already ruled out for problem 647? See the chapter on problem 647.

What our research shows

I ran a computer search on each problem. Each result has its checked scope and its open items.

Problem 64: no counterexample up to 22 vertices (Sonny, 2026). What this is: no graph with minimum degree at least 3 on up to 22 vertices lacks a cycle of length 4 or 8. Status COMPUTED. Two recent preprints report larger ranges (arXiv:2609.04686, 23 vertices; arXiv:2609.28594, 40 vertices). Still open: the conjecture beyond the finite search stated here. Checked by: exhaustive generation with nauty's geng; a separate exhaustive search of our own up to 15 vertices; the generator logs re-read for completeness.

Problem 647: no example up to 10^{14} (Sonny, 2026). What this is: no n with $24 < n \leq 10^{14}$ satisfies the condition, as a replication of searches that the forum already reports to much larger bounds. Status COMPUTED. Scope and open item: this replicates part of the forum's reported range. Still open here: n above 10^14. Checked by: the residue lemma (a rule that rules out whole families of numbers by their remainder on division by a fixed number) recomputed here; a direct computation of every n up to 10^8, done twice by separately written programs; an arithmetic scan up to 10^9; the logs of the C sieve for the range above that re-read for completeness.

Where to go next. Why is the problem 647 result only a replication? See the chapter on problem 647.

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What's in this book

Counts are computed from the book's statement file (its list of results: 2 results, each with one status).

Status	Count
Proved and checked in Lean	0
Written proofs	0
Computed or certified (exact arithmetic, exhaustive search, or re-read logs)	2
Conditional	0
Open	0

Item	Count
Chapters	2 (problems 64 and 647)
Statement files	2
Written check recipes	3, plus the fingerprint list
Certificate and data files	19 in 2 folders (problem 64: 3; problem 647: 16)
Lean source files	0

<!--/BOX-->

Yesterday, today, tomorrow

Yesterday. Paul Erdős with András Gyárfás (problem 64) and with John Selfridge (problem 647) posed these questions, and Thomas Bloom's erdosproblems.com gathers them with the earlier work. For problem 64, the exhaustive computer searches listed there, by Markström, Nowbandegani and Esfandiari, Tranquilli and sallerk, are for cubic, bipartite or small-diameter graphs; Balaji, Garcia, and Ducoffe and Dumitru report searches of every graph with minimum degree at least 3, to 31, 23 and 40 vertices. For problem 647, Dutta, Alexeev, Kitamura, Hughes and others on the forum had reduced the problem and searched much larger ranges.

Today. For problem 64, I found that every connected graph with minimum degree at least 3 on 4 to 22 vertices has a cycle of length 4 or 8, using nauty's geng restricted to that class and a separate exhaustive search of our own up to 15 vertices; the search covers all graphs with minimum degree at least 3, not just cubic, bipartite or small-diameter ones, and stops at 22 vertices; orders 16 to 22 rest on the pruned geng build, which is not published. Preprints report larger ranges (arXiv:2609.04686, 23 vertices; arXiv:2609.28594, 40 vertices). For problem 647, no n with $24 < n$ up to 10^{14} satisfies the condition; this is a replication of a smaller range than the forum reports.

Tomorrow. For problem 64, order 23 and beyond, in step with what the two preprints report (all graphs to 40 vertices, bipartite to 66, cubic to 48). For problem 647, the separate code and logs give a cross-check on the forum's searches. None of these settles its problem.

Where to go next. Which claims need a reader and not a program? See the verification guide and the statements.

Check it yourself

What ships: the generator logs for problem 64 (orders 4 to 22), the residue-class list and the sieve logs for problem 647, a SHA-256 fingerprint (a 64-character code computed from a file; change one bit and the code changes) for each file, and a written recipe for each check. With these a reader can recheck that every part of each search finished and that the counts and ranges add up. What is not shipped, and is available on request: the geng build with our pruning, which produced the problem 64 logs for orders 16 to 22, and the C source of the sieve and of the brute force for problem 647. The recipes check the bookkeeping of the logs; they do not rerun the searches. A recipe gives the exact file format, the exact condition to test, the expected answer and a standard tool or a few lines of code that any reader can write. The recipes are in the verification guide, which starts with the fingerprint check. To compare file fingerprints, any SHA-256 tool will do:

```
sha256sum -c SHA256SUMS.txt  # Linux
shasum -a 256 -c SHA256SUMS.txt   # macOS
CertUtil -hashfile <file> SHA256
```

The upper command is run inside the certificates folder on Linux (macOS: the shasum line); the lower one does the same for one file on Windows. No Lean proof is shipped in this book.

How I direct this work, and my own tools, are not part of this book. The files listed above ship; the files named there as not shipped are available on request.

About this research: https://gocheckyourself.org/about/#how

Where to go next

What does a pass of each check show, and what does it leave open? See the verification guide. The chapters:

- Problem 64: cycles of power-of-two length
- Problem 647: the divisor function and the next integer

We stand on the shoulders of giants. The credits page lists the work this builds on.

Go check yourself.

Have a question about this? Ask it

Erdős #64: cycles of power-of-two length

Erdős #64: cycles of power-of-two length

The question

Erdős and Gyárfás asked whether every finite graph (dots, called vertices, joined by lines) with minimum degree (the fewest lines at any vertex) at least 3 contains a cycle (a closed loop of distinct vertices) whose length is a power of two (4, 8, 16, 32, ...). The problem page (erdosproblems.com/64) says they "believed the answer must be negative", and that for every r there must be a graph of minimum degree at least r with no such cycle. The page lists a prize of $1000 and does not say who offered it. Liu and Montgomery proved that the answer is yes when the minimum degree is larger than some absolute constant.

Why it matters

Hong Liu and Richard Montgomery showed the answer is yes once the minimum degree is large enough, so the open part is sparse graphs, such as those with minimum degree 3; a counterexample (an example that would show the claim is false) would have to be found among them.

What answering it would help with

A counterexample would show that the large-minimum-degree theorem needs its degree condition; the stated coverage of an exhaustive search guides the planning of a larger search or a proof.

What our research shows

We stand on the shoulders of giants. Every earlier result named here is theirs; the credits page lists them with references.

What was already checked by computer

As listed on the problem page and its forum:

- Markström: a cubic counterexample (every vertex of degree exactly 3) needs at least 30 vertices.
- Nowbandegani and Esfandiari: a bipartite counterexample (one whose vertices split into two groups, with every line joining the two groups) needs at least 32 vertices.
- Nowbandegani, Esfandiari, Haghighi and Bibak: a cubic claw-free counterexample (no vertex has three neighbours with no lines among them)

needs at least 114 vertices.
- sallerk (forum): no cubic bipartite counterexample on at most 62 vertices; general cubic graphs searched to 34 vertices.
- Results for whole families of graphs, proved by hand (Alfaiz's list on the forum), and Temeller's searches for graphs of diameter at most 3 (any two vertices are at most 3 steps apart).
    - **Avery Carr**, arXiv:2508.19302: every graph of diameter 2 with minimum degree at least 3 has a cycle of length 4 or 8.
    - **Avery Carr**, arXiv:2605.22844: in a smallest counterexample, at least 4/7 of the vertices have degree 3. Ducoffe and Dumitru (arXiv:2609.28594) raise this to more than 2/3.
    - **Michael C. Burkhart**, arXiv:2611.05060: Cayley graphs of odd order satisfy the conjecture (so do Cayley graphs of finite nilpotent groups).
    - **Julius Tranquilli**, arXiv:2608.02675: a cubic bipartite counterexample needs at least 60 vertices (every cubic bipartite graph on at most 58 vertices has a cycle of length 4, 8 or 16).

As far as the sources we cite show, the earlier exhaustive searches are restricted to cubic, bipartite or small-diameter graphs. The search described here covers all graphs with minimum degree at least 3 and stops at 22 vertices. Two recent preprints report searches over all graphs up to 23 and up to 40 vertices (see below).

What was checked here

The question answered here. Is there a graph with minimum degree at least 3 and at most 22 vertices that has no cycle whose length is a power of two?

Result. Every connected graph on at most 22 vertices with minimum degree at least 3 contains a cycle of length 4 or 8. A counterexample to the conjecture has no cycle of length 4 or 8, so there is none on at most 22 vertices. This covers cubic and non-cubic graphs alike. Order 23 (graphs with 23 vertices) is reported done in arXiv:2609.04686 (a preprint; its abstract reports a SAT-based search with DRAT proofs; a SAT solver is a program that settles yes-or-no logic puzzles, and DRAT proofs are logs that let others check its answers). The 23-vertex report is credited to that preprint.

Three reductions (proved here)

The three reductions are called, in order, the connected reduction, the edge-minimal reduction and the degree cap.

Connected reduction. A component of a counterexample is a counterexample with no more vertices. So it is enough to look at connected graphs.

Edge-minimal reduction. Let uv be an edge whose two ends both have degree at least 4. Deleting uv keeps the minimum degree at least 3, and deleting an edge creates no cycle.

Repeat until no such edge is left. With the connected reduction: if a counterexample on at most N vertices exists, then a connected one exists on at most N vertices in which the vertices of degree at least 4 are pairwise non-adjacent.

Degree cap. Let G have minimum degree 3 and no 4-cycle, and let v have degree d. Each neighbour u of v has at most one other neighbour inside N(v), because two would close a 4-cycle through v. So u has at least one neighbour outside the closed neighbourhood of v. No two neighbours of v share such an outside vertex, because that would close a 4-cycle. Hence n is at least 2d + 1, so d is at most ⌊(n−1)/2⌋.

The search

The generator is geng from nauty 2.8.9 (McKay and Piperno; nauty is a standard program for working with graphs, and geng lists graphs of a chosen kind). It was called as shown below, with a pruning step that restricts it to the class described next. The pruning program is not part of this book.

```
geng -c -f -d3 -D<cap> n res/mod
```

In words: connected graphs, no 4-cycle, minimum degree at least 3, maximum degree at most the cap, on n vertices, with the search split into mod independent parts and the part number res taken. The output of a complete run for order n is every connected graph on n vertices, up to isomorphism (graphs that are the same with the vertices renamed count once), with minimum degree at least 3, no 4-cycle, no 8-cycle and its vertices of degree at least 4 pairwise non-adjacent. By the connected and edge-minimal reductions, an empty output at every order up to N shows there is no counterexample on N vertices or fewer.

orders	parts	degree cap	outcome
4 to 18	one run per order	none (the trivial cap n − 1)	every run reports 0 graphs generated
19	8	9	all 8 parts: 0 graphs
20	32	9	all 32 parts: 0 graphs
21	128	10	all 128 parts: 0 graphs
22	512	10	all 512 parts: 0 graphs

The caps are the bound the degree cap gives (⌊(n−1)/2⌋). The cost was about 1,100 core-seconds (seconds of one processor core) at order 19, 7,000 at order 20, 46,000 at order 21 and 322,000 (89 core-hours) at order 22. Each order costs about 6 to 7 times the one before.

Cross-checks

- At order 24 with maximum degree 3, the search finds exactly 4 graphs. That matches the four 24-vertex cubic graphs with no 4- or 8-cycle attributed to Markström (as reported by sallerk; we did not check this against his paper). Each of the four has a 16-cycle. They are listed in the file of 24-vertex graphs in the certificates folder, and a short program of our own confirms that each is 3-regular, has no 4- or 8-cycle and has a 16-cycle.
- A separate exhaustive search of our own, written without nauty, finds no graph for orders 4 to 15. The check recipe in the verification guide says what class it searches and what it reports.

What this does not show

- Order 23 and above are not covered by our search. Order 23 is reported in arXiv:2609.04686 (preprint).
- The shipped logs are the last lines geng printed for each part. They show that every part finished and found nothing. They cannot show that the geng binary that ran was built to prune exactly the class described above. That rests on the build, which we do not ship as a binary or as source.
- Our own separate search reaches order 15, not 22.
- This does not settle the problem and does not meet any prize condition.

Yesterday, today, tomorrow

Yesterday. Erdős and Gyárfás asked whether every finite graph with minimum degree at least 3 has a cycle whose length is a power of two, and, according to the problem page, "believed the answer must be negative"; Liu and Montgomery proved yes for large enough minimum degree. As far as the sources we cite show, the earlier exhaustive searches (Markström; Nowbandegani and Esfandiari; Nowbandegani, Esfandiari, Haghighi and Bibak; Tranquilli; sallerk) were for cubic, bipartite or small-diameter graphs (the claw-free bound is for cubic claw-free graphs; the small-diameter searches are Temeller's), and the lower bounds they give reach 30, 32, 60 and 114 vertices for those families (sallerk's searches reach 34 vertices for cubic graphs and 62 for cubic bipartite graphs). Two recent preprints report larger ranges: arXiv:2609.04686 reports that every such graph on at most 23 vertices has a cycle of length 4 or 8, and arXiv:2609.28594 reports the same check for all graphs on at most 40 vertices, all bipartite graphs on at most 66 and all cubic graphs on at most 48. Our search covers 4 to 22 vertices.

Today. I generated, with nauty's geng restricted to this class, every connected graph on 4 to 22 vertices with minimum degree at least 3 and no 4-cycle, no 8-cycle and its vertices of degree 4 or more pairwise non-adjacent, and found none; with the three reductions proved in this chapter, this shows every such graph on at most 22 vertices has a cycle of length 4 or 8. The search covers every connected graph with minimum degree at least 3, cubic or not, stops at 22 vertices, and cost about 89 core-hours at

order 22. At order 24 with maximum degree 3 it finds exactly the four graphs attributed to Markström, as reported by sallerk. A separate exhaustive search of our own reaches order 15, and the generator logs were re-read for completeness.

Tomorrow. Our search continues on an ordinary home computer, and we expect to report more substantial results in future. Each order costs roughly 6 to 7 times the one before. The two preprints above report much larger ranges (all graphs to 40 vertices, bipartite to 66, cubic to 48). The problem stays open.

Check it yourself

The verification guide has two recipes for this problem: a bookkeeping check of the generator logs for orders 4 to 22, and a small-order search with the four 24-vertex graphs.

Where to go next

- How would a counterexample to the divisor-function problem be found, and why can a search reach 10^{14}? See the chapter on problem 647.
- What does a pass of the log check show, and what does it not? See the verification guide.

We hope it helps others find more results and keeps their curiosity going. We'd love to collaborate with anyone.

Erdős #647: m + τ(m) and the next integer

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The question

Problem #647 (erdosproblems.com/647), due to Erdős and Selfridge: is there an $n > 24$ with max over $m < n$ of $(m + \tau(m)) \le n + 2$? Here $\tau(m)$ is the number of divisors of m. Erdős offered 25 pounds for an example. Writing $m = n - k$, the condition is

$\tau(n - k) \le k + 2$ for every $k = 1, \ldots, n - 1$. $(*)$

The numbers $n = 1, 2, 3, 4, 5, 6, 8, 10, 12$ and 24 satisfy it.

Why it matters

It asks whether the divisor function can stay small enough, just below an integer, for all smaller numbers at once; the forum has ruled out every n up to very large bounds,

so any example would be very large.

What answering it would help with

An example, or a proof that none exists, would determine whether the divisor function can stay within the stated limits on runs of consecutive integers; separate replications give cross-checks on the forum's searches.

What our research shows

We stand on the shoulders of giants. This chapter is a replication of work already on the forum, in a smaller range than the forum reports. The credits page lists the sources.

What the forum reports

As read through a page reader; the fragments were not checked against the raw pages, and handles are as shown there.

- Dutta (January 2026): divisibility filters (rules that rule out whole families of n) ending in n = 2520N. Alexeev: a repair of one step, and the observation that 2520N − 1 is then prime. Kitamura: (n − 3)/3 is prime for n > 84.
- Hughes (May to June 2026): a reduction to 41 residue classes (numbers grouped by their remainder on division by 46189) modulo 46189, density bounds, a search to about 6.16×10^{17} and a certificate (a file that lets a reader check the claim) for no solution with $24 < n \le 615{,}736{,}321{,}200{,}000$. With bentrd, a frontier of about 9.17×10^{18} (second-hand: no bentrd comment was visible on the page as read).
- Idén (June 2026): no solution in $(24, 10^{12}]$; the comment cites an earlier bound of 10^{10} in OEIS A087280 (the On-Line Encyclopedia of Integer Sequences, a public catalogue of number sequences).

- IbbyMian: a Lean 4 proof (Lean is a program that checks every step of a proof against fixed rules) to 10^8. eerot (October 2026): a kernel-checked exclusion (every step passed Lean's small core checker) to 10^13.
- Mian and Siddique (arXiv:2608.17880, August 2026): a Lean 4 proof, using only the three standard axioms, that no n with $24 < n \le 10^9$ satisfies the condition. The range of this chapter ends at 10^{14} and is computed; no Lean proof ships with it.
- veljjanoski (September 2026): GPU searches (run on graphics cards, which do many small calculations at once) to 10^{20} and then 10^{22}, without a certificate.

Our search stops at 10^14. It lies inside what Hughes reports with a certificate and beyond eerot's Lean-checked 10^13. It is a replication, with separate code, separate proofs and a separate log, and it does not extend any of these ranges.

Result

The question answered here. Is there an n above 24, up to 10^{14}, for which m + τ(m) stays at most n + 2 for every m below n?

No n with $24 < n \le 10^{14}$ satisfies (∗). Together with the brute force (trying every number) up to 6,350,464 (the threshold of the residue lemma below), the solutions in $[1, 10^{14}]$ are exactly the ten numbers above. A direct computation of every $n \le 10^8$, which does not use the residue lemma, gives the same ten numbers: it was done in C and done again with a separately written program.

The argument (all elementary)

Lemma 1 (lower bound). Let S be a finite set of primes, and write m = d·c where d is the part of m made of primes in S and c is coprime to them. Then τ(m) = τ(d)·τ(c), because τ is multiplicative, and τ(c) = 1 if c = 1, τ(c) = 2 if c is prime, τ(c) ≥ 3 if c is composite. So knowing the exact exponents of the primes in S gives a lower bound on τ(m).

Lemma 2 (residue lemma; it recovers the reduction of Dutta and Alexeev). Let M = 2^6·3^4·5^2·7^2 = 6,350,400 and K = 64. Suppose $n \equiv r \bmod M$ and n > M + K. For each k ≤ K and each prime power p^a exactly dividing M, put t = (r − k) mod p^a. If t ≠ 0 then the exponent of p in n − k is exactly the exponent of p in t. If t = 0 it is at least a. This gives a lower bound b(r, k) for the product of (exponent + 1) over p = 2, 3, 5, 7. When all four exponents are exact, the {2,3,5,7}-part of n − k is less than M < n − k, so n − k has another prime factor and b may be doubled (Lemma 1). If b(r, k) > k + 2 for some k ≤ 64, no n in the class r satisfies (∗). The classes that survive are 2205 in number, and all of them are multiples of 2520, which matches the published reduction. They can be recomputed in a few lines; the check recipe says how.

Lemma 3 (the filter at k = 1, 2; the observation of Alexeev and Dutta on the forum, written out in elementary form). If n > 10 satisfies (∗) then n − 1 is prime and (n − 2)/2 is prime. Proof. From k = 1, τ(n − 1) ≤ 3, so n − 1 is 1, a prime or a prime square. If n were odd, n − 1 would be an even prime or the square of one, so n ≤ 5. So n is even and n − 2 is even. From k = 2, τ(n − 2) ≤ 4. If 4 divides n − 2, write n − 2 = 2^a·u with u odd; then (a + 1)·τ(u) ≤ 4 forces u = 1 and a ≤ 3, so n ≤ 10. So 4 divides n and n − 2 = 2u with u odd and 2τ(u) ≤ 4, so u is 1 or prime, and u = 1 gives n = 4. Now $n - 1 \equiv 3 \bmod 4$, and an odd square is 1 mod 4, so n − 1 is prime (this last step is Alexeev's: 2520N − 1 is 3 mod 4, so it cannot be a square). ■

Lemma 4 (soundness of the sieve). The C sieve (a program, written in C, that strikes out numbers in bulk) that produced the logs for n above 10^8 works on n = r + M·j. For each of the 2205 classes and each k ≤ 64 it removes every prime power p^e ≤ n with p ≤ 65536 from n − k exactly, so the exponent of every such p is exact. What is left, the cofactor c, has only prime factors above 65536. It is prime when it is below 65536^2, and

a deterministic Miller-Rabin test (a quick way to tell whether a number is prime) decides it otherwise (the base sets of Sorenson and Webster are deterministic far beyond 10^{14}). Lemma 1 then gives a lower bound on $\tau(n - k)$, and discarding n when that bound exceeds $k + 2$ can never discard a true solution.

Lemma 5 (finitely many k suffice). Let $T(x)$ be the largest $\tau(m)$ for $m \le x$. If $\tau(n - k) \le k + 2$ for all $k \le T(n) - 2$, then $(*)$ holds for all k, since $\tau(n - k) \le T(n) \le k + 2$ beyond that. $T(x)$ can be computed rigorously from the exponent patterns of highly composite numbers. No survivor of the sieve ever arose, so this lemma was only used on test inputs.

Coverage

The sieve ran in chunks. Its output lists, for every class and chunk, the number of candidates n and the number left after each stage. The check recipe says how to read it. A reader checks that every one of the 2205 classes appears exactly once in each chunk, that each class's candidate count equals the exact number of integers $n \equiv r \pmod{M}$ in the chunk, that the chunks are contiguous from $10^8 + 1$ to 10^{14}, and that no chunk lists a survivor.

range	candidates	survivors
$(10^8, 10^{10}]$	3,437,500	0
$(10^{10}, 10^{12}]$	343,750,000	0
$(10^{12}, 10^{13}]$	3,125,000,000	0
$(10^{13}, 3{\cdot}10^{13}]$	6,944,444,444	0
$(3{\cdot}10^{13}, 10^{14}]$	24,305,555,556	0
total	34,722,187,500	0

Every candidate was removed by the lower-bound sieve (stage 1), so the later stages (Miller-Rabin and two exact τ programs) never met a production value.

What the bundle re-establishes by running, and what it does not

- Run here with Python programs of our own: the residue lemma (the 2205 classes); a brute-force computation for $n \le 6{,}350{,}464$, which finds exactly the ten numbers; an arithmetic scan of every candidate n up to 10^9 in the 2205 classes, written independently of the C sieve, with no survivors; and the coverage check of the shipped logs. The programs are not part of this book; the recipe in the verification guide gives what to test.
- A direct computation for every $n \le 10^8$, with $\tau(m)$ for all $m \le 10^8$ from a divisor-count sieve and no use of the residue lemma: done in C (about 8 seconds on one

core) and again by a separately written program (about 16 seconds). Both find exactly the ten numbers 1, 2, 3, 4, 5, 6, 8, 10, 12 and 24.
- Not re-run: the C sieve itself for n above 10^9. For that range the bundle ships the logs and checks their bookkeeping. The C sources are available on request.

- The search is finite. It says nothing about n above 10^14.

Yesterday, today, tomorrow

Yesterday. Erdős and Selfridge asked whether any n above 24 has the property that m plus the number of divisors of m stays at most n plus 2 for every m below n. On the forum, Dutta, Alexeev and Kitamura found filters that reduce the search; Hughes reduced it to 41 residue classes and searched to about 6.16 x 10^{17} with a certificate; Idén reported no solution up to 10^{12}; IbbyMian and eerot gave Lean-checked exclusions to 10^{8} and 10^{13}; and veljjanoski searched to 10^{22} without a certificate.

Today. This chapter is a replication, in a smaller range than the forum reports, with separate code, proofs and logs. I recomputed the residue lemma (2205 classes, all multiples of 2520), found by brute force that the solutions up to 6,350,464 are exactly the ten numbers 1, 2, 3, 4, 5, 6, 8, 10, 12 and 24 (a direct computation up to 10^{8}, done twice by separately written programs, finds the same ten), and ran an arithmetic scan to 10^{9} with no survivors. For 10^{8} to 10^{14} the bookkeeping of the sieve logs was checked (34,722,187,500 candidates, none surviving); the sieve itself was not rerun above 10^9. It extends none of the forum's ranges and says nothing above 10^14.

Tomorrow. The separate code and logs may serve as a cross-check for people comparing searches on the forum. The elementary lemmas are short enough to reuse in other searches of this kind.

Check it yourself

The recipe for problem 647 in the verification guide recomputes the residue classes, the brute force and the scan, and checks the bookkeeping of the shipped sieve logs.

Where to go next

- How do the exhaustive searches for problem 64 reach their bound, and why do they stop at 22 vertices? See the chapter on problem 64.
- How can you check the sieve logs yourself? See the recipe for problem 647.

We hope it helps others find more results and keeps their curiosity going. We'd love to collaborate with anyone.

Try it yourself

The figures of this chapter are the interactive checks on the tools page (they need JavaScript).

Each result on its own page

- Lemma 1: the connected reduction
- Lemma 2: the edge-minimal reduction
- Lemma 3: the degree cap
- Computation 4: no graph on 4 to 22 vertices with minimum degree at least 3 lacks a cycle of length 4 or 8
- Computation 5: the cross-check at order 24
- Computation 6: the separate exhaustive search to order 15
- Lemma 7: lower bound from the divisors of known primes
- Computation 8: the residue lemma
- Lemma 9: the filter at k = 1, 2
- Lemma 10: soundness of the sieve
- Lemma 11: finitely many k suffice
- Computation 12: the solutions up to 6,350,464
- Computation 13: the solutions up to 10^8, by direct computation
- Computation 14: no example with 24 < n <= 10^14
- Computation 15: the sieve coverage
- Open question: Erdős problem 64
- Theorem: Erdős problem 64 at 23 vertices
- Open question: Erdős problem 647

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